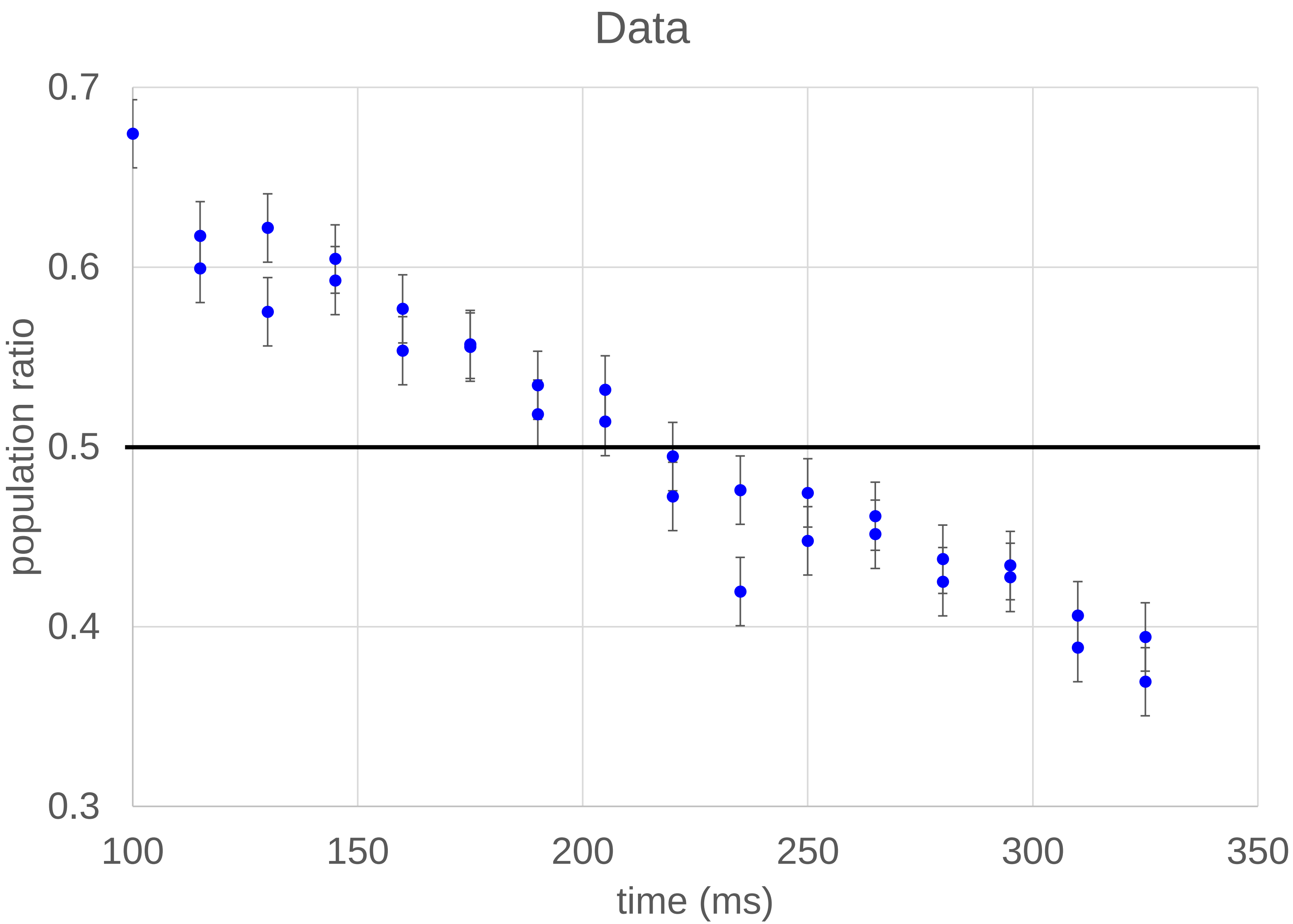
Covariance
in
Error Propagation

The covariance $\sigma_{\alpha\beta}^2$ "error" between fitting parameters α and β describes the **correlation** in their errors $\pm\Delta\alpha$ and $\pm\Delta\beta$.

→ **Correlated errors** means that if α fluctuates upwards to $\alpha + \Delta\alpha$, then β tends to fluctuates upwards as well to $\beta + \Delta\beta$.

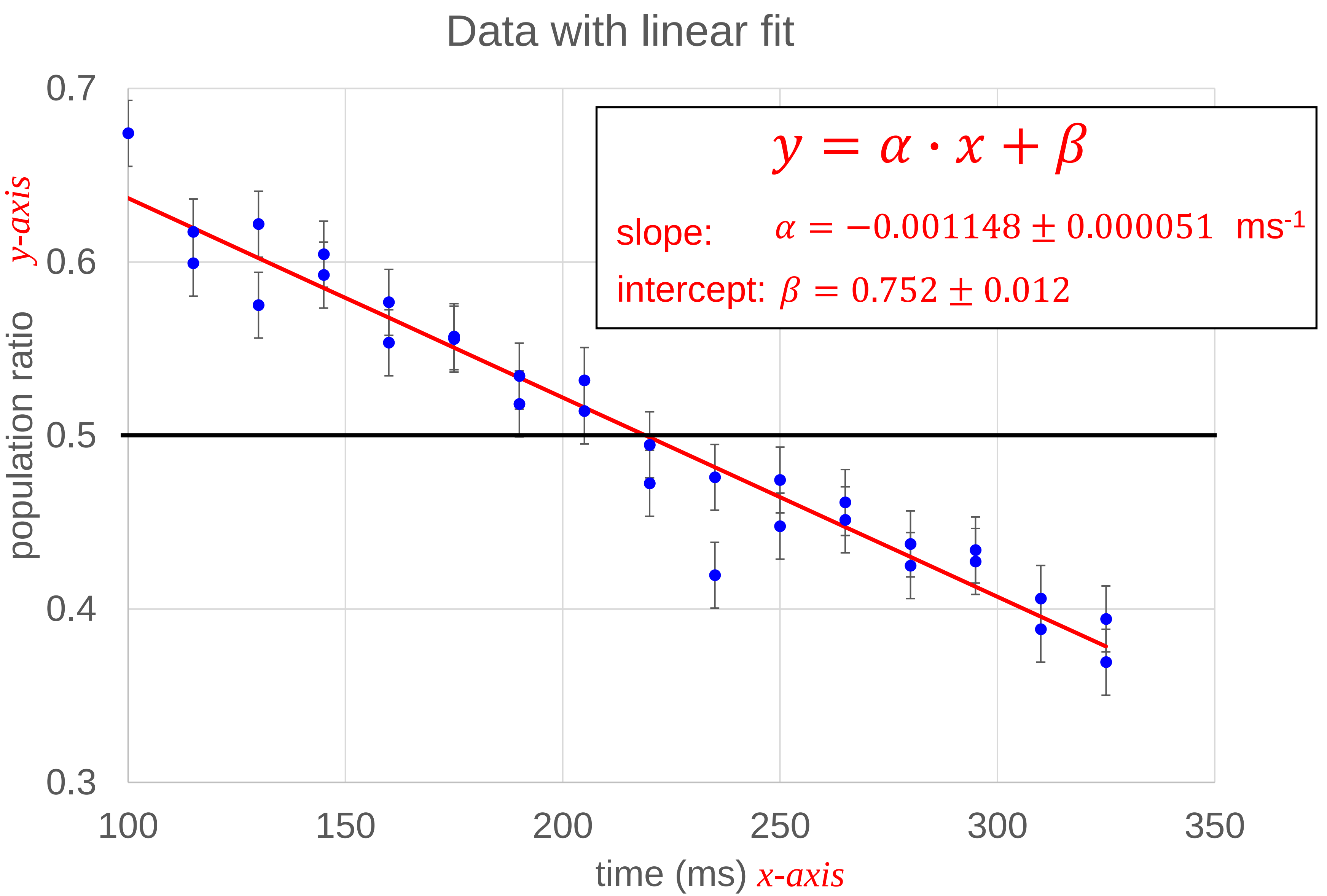
→ ***Independent errors*** means that fluctuations on α have not impact on fluctuations of β, and vice versa.

Example



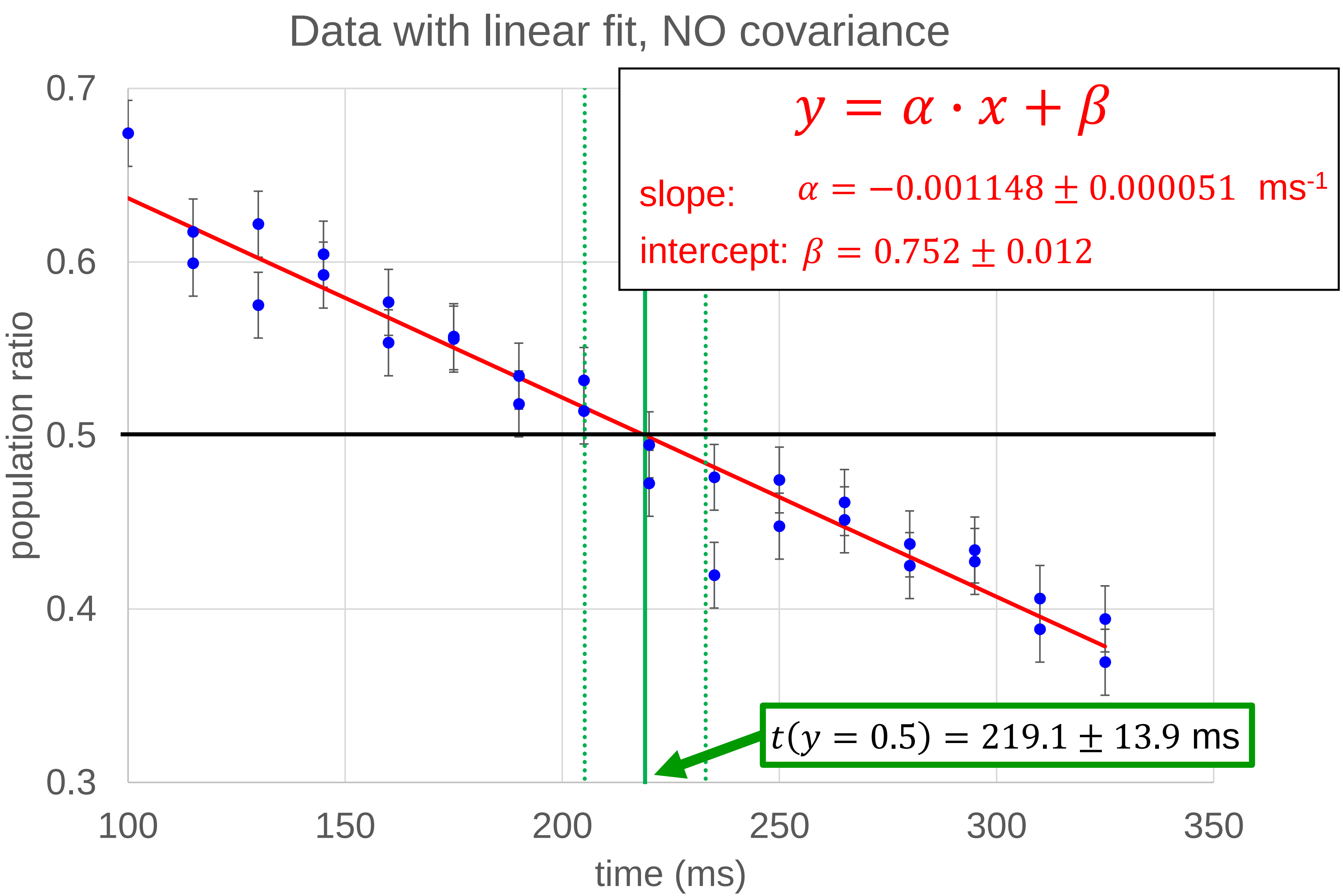
Data
population ratio
time (ms)
0.7
0.6
0.5
0.4
0.3
100
150
200
250
300
350

Example



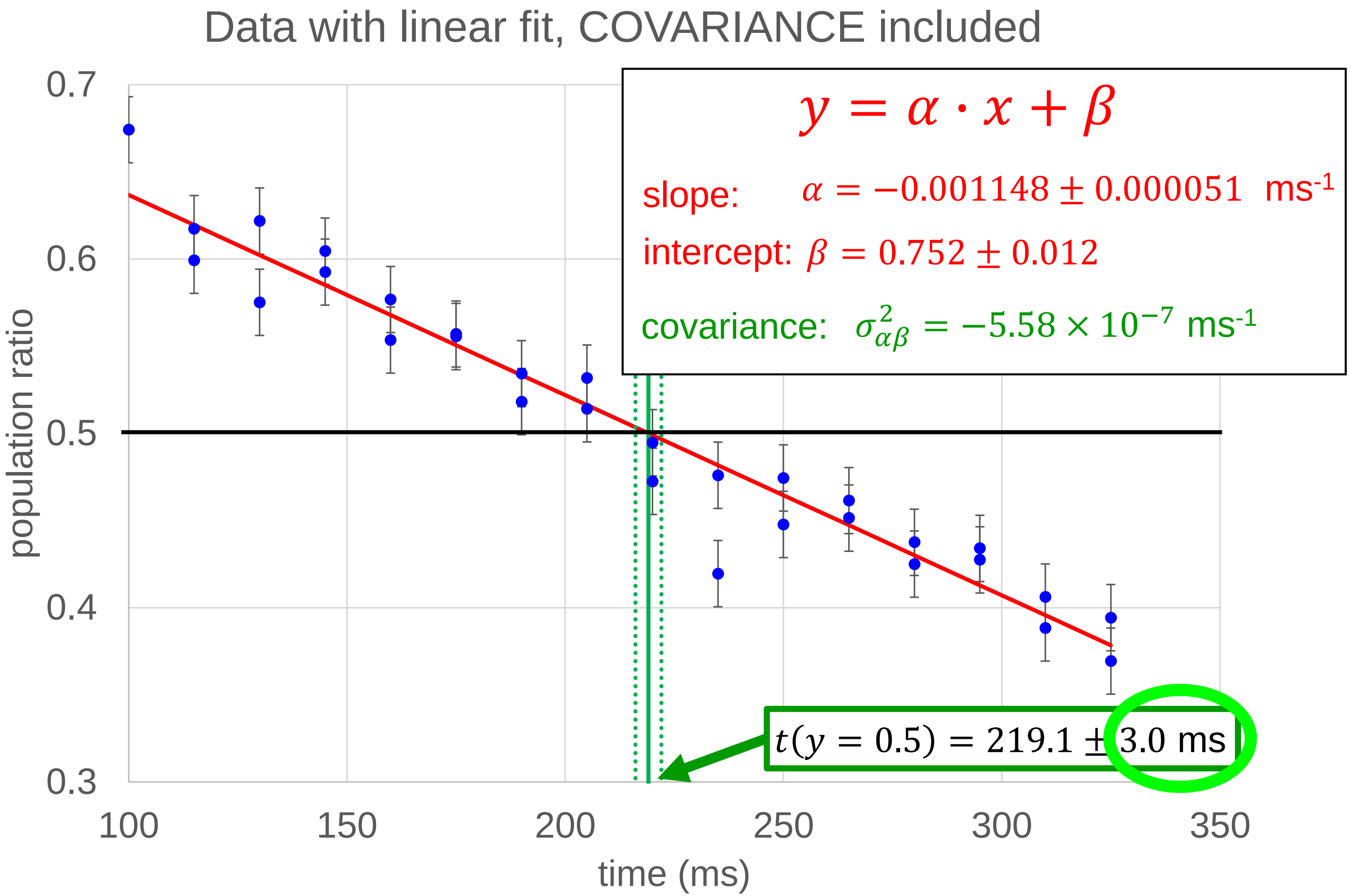
Data with linear fit
$y = \alpha \cdot x + \beta$
slope: $\alpha = -0.001148 \pm 0.000051$ ms^{-1}
intercept: $\beta = 0.752 \pm 0.012$
population ratio *y-axis*
time (ms) *x-axis*
0.7
0.6
0.5
0.4
0.3
100
150
200
250
300
350

Example



Data with linear fit, NO covariance
$y = \alpha \cdot x + \beta$
slope: $\alpha = -0.001148 \pm 0.000051$ ms^{-1}
intercept: $\beta = 0.752 \pm 0.012$
$t(y = 0.5) = 219.1 \pm 13.9$ ms
population ratio
0.7
0.6
0.5
0.4
0.3
100
150
200
250
300
350
time (ms)

Example



Data with linear fit, COVARIANCE included
$y = \alpha \cdot x + \beta$
slope: $\alpha = -0.001148 \pm 0.000051$ ms^{-1}
intercept: $\beta = 0.752 \pm 0.012$
covariance: $\sigma^2_{\alpha\beta} = -5.58 \times 10^{-7}$ ms^{-1}
$t(y = 0.5) = 219.1 \pm 3.0$ ms
0.7
0.6
0.5
0.4
0.3
population ratio
100
150
200
250
300
350
time (ms)

Error Propagation with Covariance

If you want to calculate the error $\pm\Delta f$ on calculating $f(\alpha,\beta)$, then the correct formula is:

$$\pm\Delta f = \pm\sqrt{(\frac{\partial f}{d\alpha})^2(\Delta\alpha)^2 + (\frac{\partial f}{d\beta})^2(\Delta\beta)^2 + 2(\frac{\partial f}{d\alpha})(\frac{\partial f}{d\beta})\sigma_{\alpha\beta}^2}$$

covariance

Covariance Matrix

When you use the “curve_fit” function from the “scipy.optimize” library, Python returns the “finalParametersErrors” array.

The “finalParametersErrors” array is actually called the “covariance matrix”:

finalParametersErrors = covariance matrix = $[\sigma] = \begin{bmatrix} (\Delta\alpha)^2 & \sigma_{\alpha\beta}^2 \\ \sigma_{\alpha\beta}^2 & (\Delta\beta)^2 \end{bmatrix}$

The covariance matrix (or “error matrix”) is always symmetric.

Propagating Errors without Derivatives

If we ignore correlations between errors (i.e. assume covariance = 0), then error propagation is done with the following familiar equation:

$$\pm\Delta f = \pm\sqrt{(\frac{\partial f}{d\alpha})^2(\Delta\alpha)^2 + (\frac{\partial f}{d\beta})^2(\Delta\beta)^2}$$

$$= \pm\sqrt{(\underbrace{\frac{\partial f}{d\alpha}\Delta\alpha)}^2 + (\frac{\partial f}{d\beta}\Delta\beta)^2}$$

$$\frac{\partial f}{d\alpha}\Delta\alpha \approx f(\alpha+\Delta\alpha,\beta) - f(\alpha,\beta)$$

(definition of derivative)

$$\approx \frac{1}{2}[f(\alpha+\Delta\alpha,\beta) - f(\alpha-\Delta\alpha,\beta)]$$

Propagating Errors without Derivatives

If the derivative is complicated to derive, then you can just evaluate $f(\alpha, \beta)$, $f(\alpha \pm \Delta\alpha, \beta)$, and $f(\alpha, \beta \pm \Delta\beta)$ and use the formula:

$$\pm\Delta f = \pm\sqrt{[f(\alpha + \Delta\alpha, \beta) - f(\alpha, \beta)]^2 + [f(\alpha, \beta + \Delta\beta) - f(\alpha, \beta)]^2}$$

or

$$\pm\Delta f = \pm\sqrt{[f(\alpha - \Delta\alpha, \beta) - f(\alpha, \beta)]^2 + [f(\alpha, \beta - \Delta\beta) - f(\alpha, \beta)]^2}$$

or

$$\pm\Delta f = \pm\sqrt{\begin{array}{l}\frac{1}{4}[f(\alpha + \Delta\alpha, \beta) - f(\alpha - \Delta\alpha, \beta)]^2 + \cdots \\ \ldots + \frac{1}{4}[f(\alpha, \beta + \Delta\beta) - f(\alpha, \beta - \Delta\beta)]^2\end{array}}$$