Thursday, August 31, 2017

Basic Review of time-independent E&M
(in vacuum)

E-field satisfies:

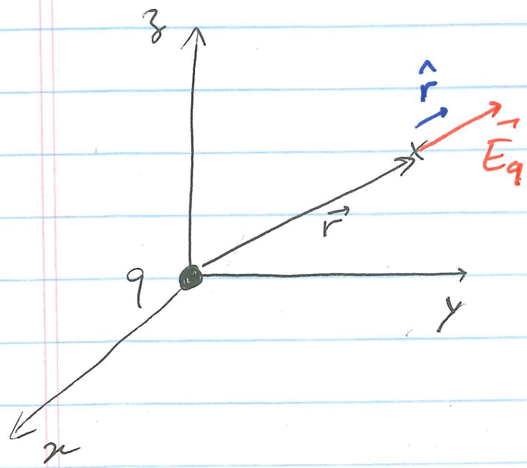
- $$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho(\vec{r})}$$ Gauss's Law

where $\varepsilon_0 = 8.85\times10^{-12}\ C^2/N\cdot m^2$
= permittivity of free space

ρ = charge density in $\frac{\text{Coulombs (C)}}{m^3}$

- $$\boxed{\vec{\nabla}\times\vec{E} = 0}$$

E-field of a point charge: $\vec{E}_q = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}|^2}\hat{r}$



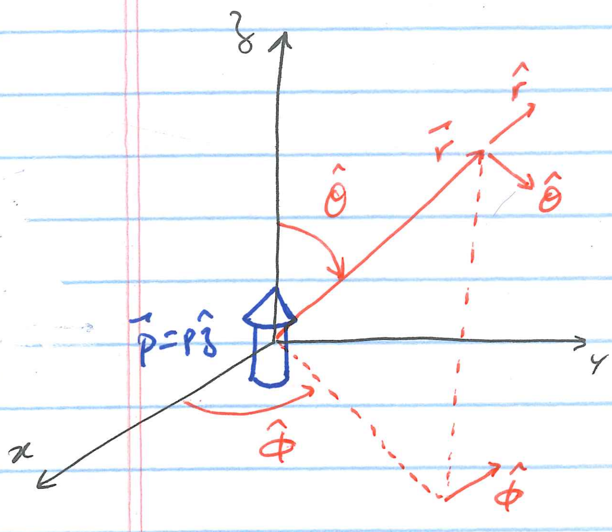
charge density of a point charge q:

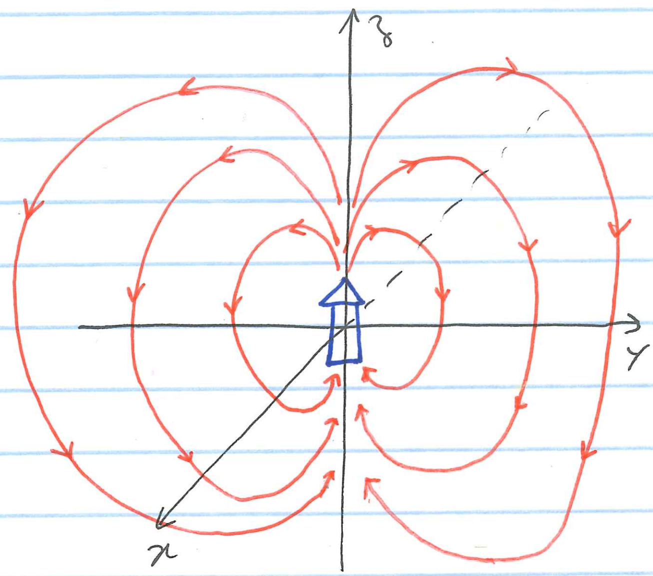
$$\rho(\vec{r}) = q\,\delta^3(\vec{r}-\vec{r}_0)$$
$$= q\,\delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

Electric field of an ideal electric dipole $\vec{p}$:

$$\vec{E}_{dipole} = \frac{|\vec{p}|}{4\pi\varepsilon_0}\,\frac{1}{r^3}\,\underbrace{\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}_{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}$$

"coordinate free" version



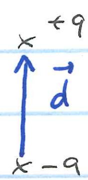


cylindrically symmetric

(also spherical symmetry)

$$\vec{p} = \int \vec{r}\,\rho(\vec{r})\,d^3r$$

$$= q\vec{d} \quad \text{for}$$

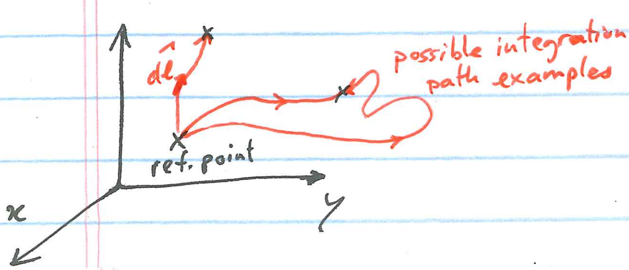


Potential formulation of Electrostatics

(derived from $\vec{\nabla}\times\vec{E} = 0$)

Electric potential: $V(\vec{r}) = -\int_{\text{reference point}}^{\vec{r}} \vec{E}(\vec{r}\,')\cdot d\vec{l}\,'$

frequently chosen at $\vec{r}$ = infinity



does not depend on integration path

⚠ but does depend on the reference point

Potential difference does not depend on the ~~refere~~ reference point.

$$\Delta V = V(\vec{r}_b) - V(\vec{r}_a)$$

$$= -\int_{\text{ref. point}}^{\vec{r}_b} \vec{E}\cdot d\vec{l} - \left(-\int_{\text{ref. point}}^{\vec{r}_a} \vec{E}\cdot d\vec{l} \right)$$

$$= -\int_{\text{ref. point}}^{\vec{r}_b} \vec{E}\cdot d\vec{l} - \int_{\vec{r}_a}^{\text{ref. point}} \vec{E}\cdot d\vec{l} = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}\cdot d\vec{l}$$

note: gradient

$$\vec{\nabla} V = \left(\lim_{\Delta x \to 0} \frac{V(\vec{r}+\Delta x\,\hat{x}) - V(\vec{r})}{\Delta x}, \lim_{\Delta y \to 0} \frac{V(\vec{r}+\Delta y\,\hat{y}) - V(\vec{r})}{\Delta y}, \lim_{\Delta z \to 0} \frac{V(\vec{r}+\Delta z\,\hat{z}) - V(\vec{r})}{\Delta z} \right)$$

$$= \lim_{\Delta x, \Delta y, \Delta z \to 0} \left(\underbrace{-\frac{1}{\Delta x}\int_{\vec{r}}^{\vec{r}+\Delta x\hat{x}} \vec{E}\cdot d\vec{l}}_{-E_x}, \underbrace{-\frac{1}{\Delta y}\int_{\vec{r}}^{\vec{r}+\Delta y\hat{y}} \vec{E}\cdot d\vec{l}}_{-E_y}, \underbrace{-\frac{1}{\Delta z}\int_{\vec{r}}^{\vec{r}+\Delta z\hat{z}} \vec{E}\cdot d\vec{l}}_{-E_z} \right)$$

$$= -\vec{E} \quad \Rightarrow \quad \boxed{\vec{E} = -\vec{\nabla} V}$$ differential formulation

Potential of a point charge: $V_q(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\vec{r}|}$
(ref. point at infinity)

Potential of an ideal dipole: $V_{dipole}(\vec{r}) = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^2} = \frac{|\vec{p}|\cos\theta}{4\pi\varepsilon_0 r^2}$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0}$$

$$\vec{\nabla}\times\vec{E} = 0$$

(i.e. $\vec{E} = -\vec{\nabla}V$)

$$\Rightarrow \quad \boxed{\vec{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}}$$

poisson's equation

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

<u>Energy</u>: Electrostatic potential energy ($\Delta U = q\Delta V$)

$$U = \frac{\varepsilon_0}{2}\int_{\text{all space}} |\vec{E}|^2 \, d^3r$$

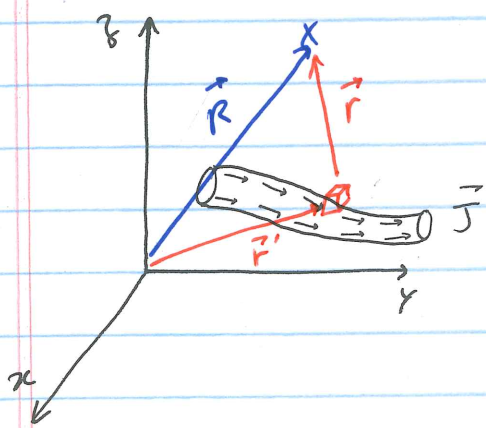
The <u>magnetic field</u> satisfies:

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0}$$ "no magnetic monopoles" law

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}}$$ Ampère's Law

<u>Biot-Savart Law</u>: $\vec{B}(\vec{R}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\vec{J}(\vec{r}\,')\times\hat{r}}{r^2}\, d^3r'$

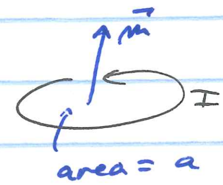
$\vec{r} = \vec{R} - \vec{r}\,'$



where $\mu_0 = 4\pi\times10^{-7}\ N/A^2$
= permeability of free space

$\vec{J}(\vec{r})$ = current density (A/m^2)

Magnetic field of a magnetic dipole: $\vec{m} = I\,\vec{a}$



$$\vec{B}_{dipole} = \frac{\mu_0 |\vec{m}|}{4\pi} \frac{1}{r^3} \Big(\underbrace{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}_{3(\hat{m}\cdot\hat{r})\hat{r} - \hat{m}} \Big)$$

$\hookrightarrow$ same form as $\vec{E}$-field of an electric dipole!

Magnetic Vector Potential $\vec{A}$:

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$$ (definition) $\leftarrow$ does not fully specify $\vec{A}$

$$\boxed{\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J}}$$ potential formulation of Ampère's law $\leftarrow$ if we choose $\vec{A}$ such that $\vec{\nabla}\cdot\vec{A} = 0$

$$\vec{A}(\vec{R}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')\, d^3r'}{|\vec{r}|}$$

for a magnetic dipole: $\vec{A}_{dipole} = \dfrac{\mu_0}{4\pi} \dfrac{\vec{m} \times \hat{r}}{r^2}$

Magnetostatic Energy:

$$U = \frac{1}{2}\frac{1}{\mu_0} \int_{\text{all space}} |\vec{B}|^2\, d^3r$$

(we will see this next week)

Lorentz Force Law:

$$\boxed{\begin{aligned} \vec{F} &= \text{force on a charge } q \\ &= q\vec{E} + q\vec{v} \times \vec{B} \end{aligned}}$$

(does not change when $\vec{E}$ & $\vec{B}$ are time-dependent)

Fundamental Vector Integral Theorems

$$\int_{\vec{r}_a}^{\vec{r}_b} (\vec{\nabla} f) \cdot d\vec{l} = f(\vec{r}_b) - f(\vec{r}_a)$$

"gradient theorem"

$$\int_{\text{volume}} \underbrace{(\vec{\nabla} \cdot \vec{F})}_{\text{divergence}} \, d^3r = \oint_{\substack{\text{bounding} \\ \text{surface}}} \vec{F} \cdot d\vec{s}$$

Gauss's / Green's theorem

or divergence theorem

$$\int_{\text{surface}} \underbrace{(\vec{\nabla} \times \vec{F})}_{\text{curl}} \cdot d\vec{s} = \oint_{\substack{\text{bounding} \\ \text{line}}} \vec{F} \cdot d\vec{l}$$

Stoke's theorem

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \iff \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$$

Ampère's law

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \iff \oint_{\substack{\text{bounding} \\ \text{surface}}} \vec{E} \cdot d\vec{s} = \frac{q_{enclosed}}{\varepsilon_0}$$

Gauss's law

Tuesday quiz (5-10 minutes)