Thursday, September 7, 2017

$\vec{\nabla}\times\vec{E} = 0$ (electrostatics) $\rightarrow$ $\underbrace{\vec{\nabla}\times\vec{E} = -\frac{d}{dt}\vec{B}}_{\text{Faraday's law}}$ (electrodynamics)

$\Leftrightarrow -\nabla V = E$

Faraday's law

↳ concept of potential is gone ... until later

Above physics, does not require a current loop (i.e. a wire) to be true.

Demonstrations: – Induced voltage on a scope.

– Eddy currents

Corollary: Lenz's Law

In the presence of a current loop (i.e. a wire or a conductor) ~~an~~ "induced" current will flow so as to minimize the change in magnetic flux ϕ.

↳ very useful for figuring out the sign/direction of the induced current flow (hence induced voltage).

Note: Faraday's law looks like Ampère's law for an E-field in a region with no (net) charges:

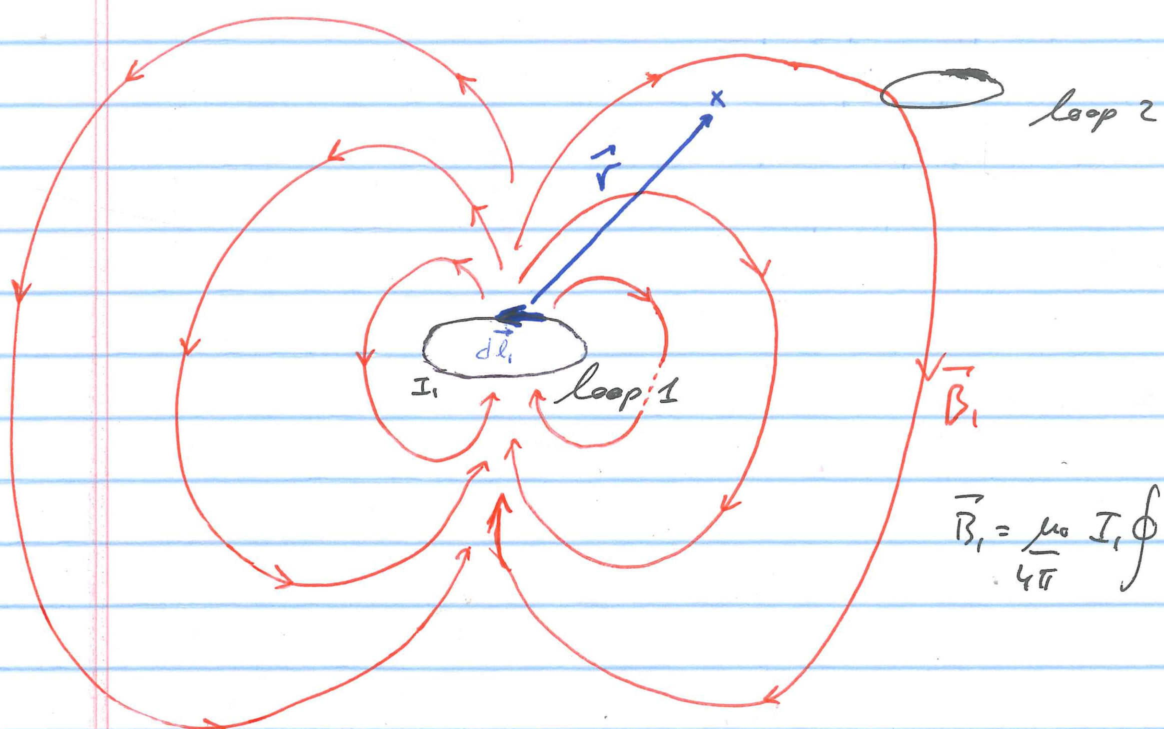
$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = \mu_0 J \end{cases} \iff \begin{cases} \vec{\nabla}\cdot\vec{E} = 0 \\ \vec{\nabla}\times\vec{E} = \underbrace{-\frac{d\vec{B}}{dt}}_{\text{source term} \equiv \text{"current"}} \end{cases}$$

$\Rightarrow$ you can apply magnetostatics calculation methods to compute the E-field produced by a changing B-field in the regions where there are no charges.

Inductance

a) Mutual Inductance

consider 2 current loops



$$\vec{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{l}_1 \times \hat{r}}{r^2}$$

The flux through loop 2 is $\Phi_2 = \int_{\substack{\text{loop 2} \\ \text{surface}}} \vec{B}_1 \cdot d\vec{s}_2 = M_{21} I_1$ (*)

M_{21} is a proportionality constant between Φ_2 and I_1 and is called the mutual inductance.

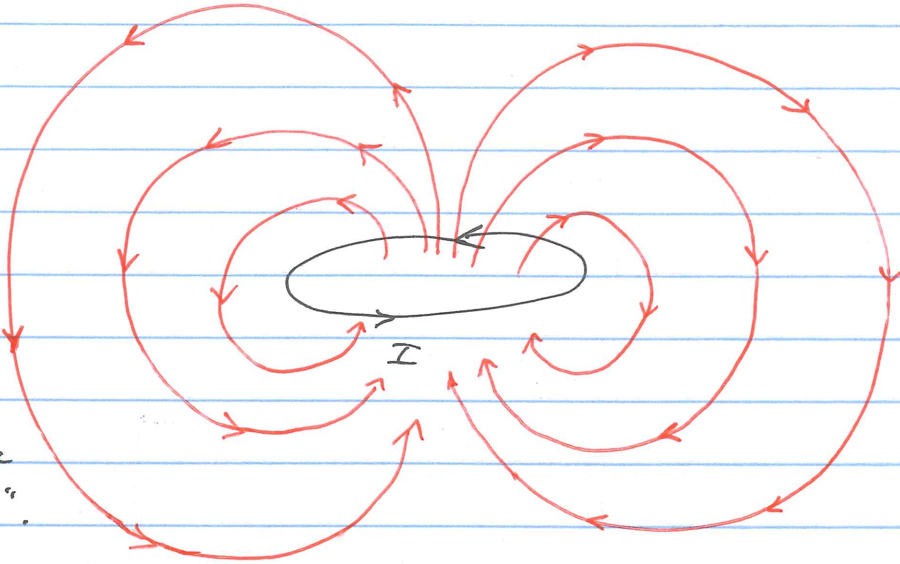
Similarly, one can define M_{12} such that $\Phi_1 = M_{12} I_2$

Properties:
- $M_{12} = M_{21} = M$
- M is a purely geometric quantity.
- For identical currents $I_1 = I_2 \Rightarrow \Phi_1 = \Phi_2$

b) Self-inductance

A single current loop can produce a flux through itself:

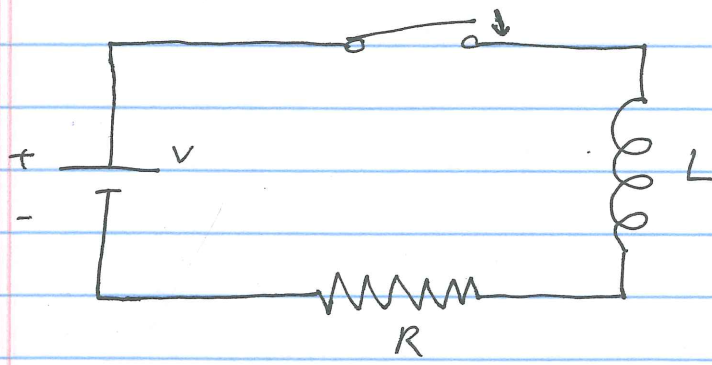
$\phi_1 = M_{11} I_1$
$\hookrightarrow \phi = L I$
where L is the self-inductance or "inductance".

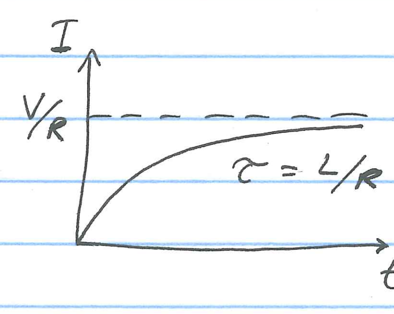


You can compute L with equation (✱) but often this is hard, unless B is simple.

c) Energy stored by an inductor

"charging up" an inductor:





EMF on the inductor: $\mathcal{E} = -\frac{d\phi}{dt} = -\frac{d}{dt}(LI)$

$$\Rightarrow \boxed{\mathcal{E} = V_L = -L\frac{dI}{dt}}$$

Power: P_L = power (into) through $L = I_L V_L = I_L \left(-L \frac{dI_L}{dt}\right)$

Energy to charge up the inductor $= E = \int_0^T P_L \, dt = -L \int_0^T I_L \frac{dI_L}{dt} dt$

$= -\frac{1}{2} L I_L^2(T)$

= work done to charge up the magnetic field in L.

$\Rightarrow \boxed{E = \frac{1}{2} L I^2}$ = Energy stored in B-field of L.

Energy of a magnetic field

we can use the energy stored in the B-field of an inductor L to derive a general formula for the energy of a B-field.

Generalization ~~note~~: Any current distribution has an associated inductance, not just a loop.

Q: How does a B-field store energy if it cannot do work?

A: $\frac{d\vec{B}}{dt} \rightarrow \vec{E} \rightarrow$ work done $\rightarrow$ energy stored

recall: magnetic flux $= \Phi = LI$ by definition

Also $\phi = \int_S \vec{B}\cdot d\vec{s} = \int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{s} = \oint_{\substack{\text{bounding}\\ \text{loop}}} \vec{A}\cdot d\vec{l}$

stoke's theorem

thus $LI = \oint_{\substack{\text{bounding}\\ \text{loop}}} \vec{A}\cdot d\vec{l}$

so $E_L = \frac{1}{2} L I^2 = \frac{1}{2} I (LI) = \frac{1}{2} I \oint_{\substack{\text{bounding}\\ \text{loop}}} \vec{A}\cdot d\vec{l}$

$I\,d\vec{l} \equiv \frac{I}{\Delta a}\, d\vec{l}\,\Delta a \equiv \int_{\text{Area}} \vec{J}\, dl\, da$

area

generalization

$\equiv \frac{1}{2} \int_{\substack{\text{volume}\\ \text{of bounding}\\ \text{loop wire}}} \vec{A}\cdot\vec{J}\, \underbrace{dl\,da}_{d^3r}$

[E_L depends on $\vec{A}$ & $\vec{J}$ at this point, not L]

but $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$ (Ampère's law)

note: $\vec{J}$ is gone!

thus $E_L = \frac{1}{2}\frac{1}{\mu_0} \int_{\substack{\text{volume}\\ \text{of current}\\ \text{distribution}}} \underbrace{\vec{A}\cdot(\vec{\nabla}\times\vec{B})}_{\vec{B}\cdot(\underbrace{\vec{\nabla}\times\vec{A}}_{\vec{B}}) - \vec{\nabla}\cdot(\vec{A}\times\vec{B})}\, d^3r$

or a larger V since $\vec{J}$ is zero there

$\underbrace{\vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{\nabla}\cdot(\vec{A}\times\vec{B})}_{\vec{B}^2 - \vec{\nabla}\cdot(\vec{A}\times\vec{B})}$

$\Rightarrow E_L = \frac{1}{2}\frac{1}{\mu_0} \int_V [\vec{B}^2 - \underbrace{\vec{\nabla}\cdot(\vec{A}\times\vec{B})}_{\int_S (\vec{A}\times\vec{B})\cdot d\vec{s}}]\, d^3r$

divergence theorem

$$\Rightarrow E_L = \frac{1}{2\mu_0}\left\{ \underbrace{\int_{\substack{\text{volume}\\ \text{of current}\\ \text{distribution}\\ \text{... or larger } V}}}_{V \to \text{ entire universe}} \vec{B}^2 \, d^3r \quad - \quad \underbrace{\int_{\substack{\text{surface}\\ \text{of current}\\ \text{distribution}\\ \text{... or larger } V}}}_{S \to \text{ surface of universe}} (\vec{A} \times \vec{B}) \cdot d\vec{s} \right\}$$

$$\underbrace{\underbrace{\underline{\text{dipole}}: \; A \propto \frac{1}{r^2}, \quad B \propto \frac{1}{r^3}}_{\frac{1}{r^5} \times r^2 \leftarrow \text{surface}}}_{\frac{1}{r^3}}$$

$\frac{1}{r^3} \to = 0$ for $r \to +\infty$

$$\Rightarrow \boxed{E_L = \frac{1}{2}\frac{1}{\mu_0}\int_{\substack{\text{all}\\ \text{space}}} \vec{B}^2 \, d^3r} \quad = \text{Energy stored in a magnetic field}$$

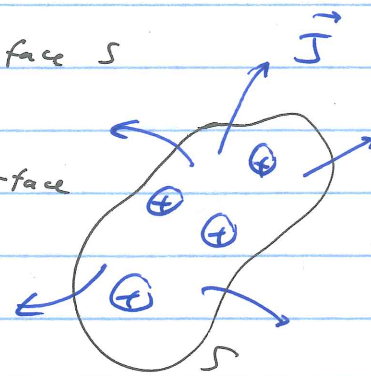
$$= \frac{1}{2} L I^2$$ (no current loop for ϕ needed)

$\Rightarrow$ Total Electromagnetic energy

$$\boxed{E_{EM} = \frac{1}{2}\varepsilon_0 \int_{\substack{\text{all}\\ \text{space}}} \vec{E}^2 \, d^3r + \frac{1}{2}\frac{1}{\mu_0}\int_{\substack{\text{all}\\ \text{space}}} \vec{B}^2 \, d^3r}$$

Conservation of charge :

Consider current flow through a closed surface S



conservation of charge : current flow through a closed surface equals the change in charge inside it.

$$\oint_S \vec{J} \cdot d\vec{s} = -\frac{\partial}{\partial t} Q_{enclosed}$$

charge density

$$\Leftrightarrow \oint_S \vec{J} \cdot d\vec{s} = -\frac{\partial}{\partial t} \int_V \rho \, d^3r$$

$$\Leftrightarrow \int_V (\vec{\nabla} \cdot \vec{J}) \, d^3r = \int_V \left(-\frac{\partial}{\partial t} \rho \right) d^3r$$

the equality is true for any $\underbrace{\text{volume / surface}}_{\text{in particular an infinitesimal one}}$ so integrands are equal

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \rho = - \vec{\nabla} \cdot \vec{J}}$$

local conservation of charge

or

continuity equation