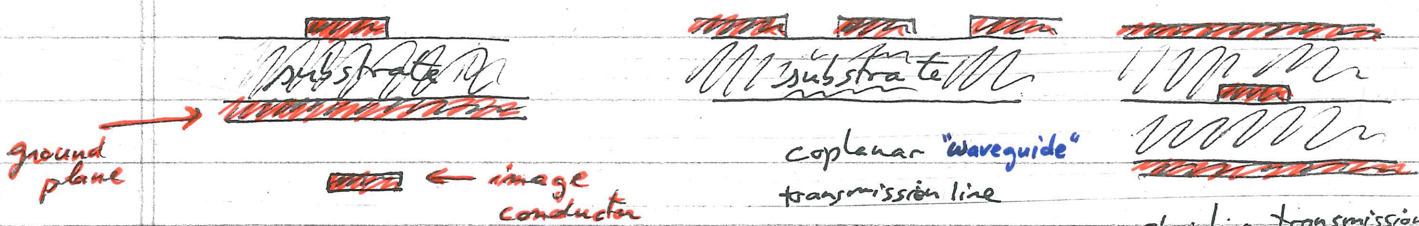
Tuesday, November 7, 2017

<u>note:</u> Any set of parallel conductors can form a transmission line. The only requirement is that the total current adds to zero.



~~Air~~ microstrip transmission line

coplanar "waveguide" transmission line

stripline transmission line

~~Question: $V_{coax} = Z_{coax} I_{coax} \Rightarrow$ Power loss $= \frac{V_{coax}^2}{Z_{coax}} = I_{coax}^2 Z_{coax}$~~

~~But there isn't any resistive heating, so what's happening?~~

~~Answer: Energy is travelling down the coax cable away from [illegible] as power loss~~

Electrodynamic Potentials

Recall Maxwell's equations:

i) $\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r},t)}{\varepsilon_0}$ ii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

iii) $\vec{\nabla} \cdot \vec{B} = 0$ iv) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(\vec{r},t) + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$

<u>Question:</u> Given $\rho(\vec{r},t)$ and $\vec{J}(\vec{r},t)$, how do you get $\underbrace{\vec{E}(\vec{r},t) \text{ and } \vec{B}(\vec{r},t)}_{\text{6 variables to solve for}}$?

(e.g. How do you generate an EM wave?)

Electrostatics: $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\vec{r}')\,(\widehat{\vec{r}-\vec{r}'})}{|\vec{r}-\vec{r}'|^2}$

or even easier $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$ and $\underbrace{\overset{\text{3 variables}}{\vec{E}} = -\vec{\nabla}\overset{\text{1 variable}}{V}}_{\text{follows from } \vec{\nabla}\times\vec{E}=0}$

Magnetostatics: $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times(\widehat{\vec{r}-\vec{r}'})}{|\vec{r}-\vec{r}'|^2}$

3 variables

or a little easier $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\, d^3r'}{|\vec{r}-\vec{r}'|}$ and $\underbrace{\vec{B} = \vec{\nabla}\times\vec{A}}_{\text{follows from } \vec{\nabla}\cdot\vec{B}=0}$

We will look for a potential formulation of Maxwell's equations (Electrodynamics)
.... maybe this will be easier!

We still have $\vec{\nabla}\cdot\vec{B} = 0$ so $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ (1) is still valid

Maxwell's equation iii)

problem: $\vec{\nabla}\times\vec{E} \neq 0 \; \left(= -\frac{\partial\vec{B}}{\partial t}\right)$ so $\vec{E} \neq -\vec{\nabla}V$

However $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Leftrightarrow \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$

$$\Rightarrow \vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = 0$$

$$\Leftrightarrow \vec{\nabla}\times\left(\vec{E} + \frac{\partial}{\partial t}\vec{A}\right) = 0$$

↳ So there should be a "$V(\vec{r},t)$" such that $\vec{E} + \frac{\partial}{\partial t}\vec{A} = -\nabla V$

$$\Leftrightarrow \boxed{\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}} \; (2)$$

Maxwell equations ii) & iii)

How do you calculate V & $\vec{A}$? → see next lecture

<u>Let's get equations of "motion" for V & $\vec{A}$</u> (eliminate $\vec{E}$ & $\vec{B}$)

$$(2) \to i) \implies \vec{\nabla}\cdot\left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right) = \frac{\rho(\vec{r},t)}{\varepsilon_0}$$

$$\iff \boxed{\vec{\nabla}^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho(\vec{r},t)}{\varepsilon_0}} \quad (3)$$

(1) & (2) → iv)

$$\implies \underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{A})}_{\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}} = \mu_0 \vec{J}(\vec{r},t) + \mu_0\varepsilon_0 \frac{\partial}{\partial t}\left[-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right]$$

$$\iff \boxed{\underbrace{\left(\vec{\nabla}^2\vec{A} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}\right)}_{\text{looks a little bit like a wave equation}} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \vec{J}(\vec{r},t)} \quad (4)$$

this does not look simpler

Equations (3) & (4) → 4 equations & 4 unknowns!
↳ simpler than Maxwell's equations (in principle)

<u>Gauge Transformations</u>

Equations (3) & (4) do not fully determine $\vec{A}$ & V.
If we do the following transformation:

$$\begin{cases} \vec{A}' = \vec{A} + \vec{\alpha} \\ V' = V + \beta \end{cases}$$

then what are the conditions on $\vec{\alpha}$ and β so that $\vec{E}$ & $\vec{B}$ remain unchanged?

If we want $\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\vec{A}'$ then $\vec{\nabla}\times\vec{\alpha} = 0$

$\Rightarrow$ there must be a $\underline{\lambda}$ such that.

$$\vec{\alpha} = \nabla\lambda$$

(in analogy with $\vec{E}$: $\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}V$)

Also plug into (2): $\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t} = -\nabla V' - \frac{\partial\vec{A}'}{\partial t}$

$$= -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} - \underbrace{\vec{\nabla}\left(\beta + \frac{\partial\lambda}{\partial t}\right)}_{\text{require } = 0}$$

we require $\nabla\left(\beta + \frac{\partial\lambda}{\partial t}\right) = 0$

$$\Rightarrow \quad \beta + \frac{\partial\lambda}{\partial t} = k(t)$$

constant in space

$$\Rightarrow \quad \beta = -\frac{\partial\lambda}{\partial t} + k(t) = \frac{\partial}{\partial t}\underbrace{\left(-\lambda + \int k(t)\right)}_{\text{rename } -\lambda}$$

$$\Rightarrow \quad \beta = \frac{\partial\lambda}{\partial t}$$

So $\vec{E}$ & $\vec{B}$ do not change so long as we perform the following gauge transformation simultaneously

$$\boxed{\begin{aligned} \vec{A}' &= \vec{A} + \vec{\nabla}\lambda \\ V' &= V - \frac{\partial\lambda}{\partial t} \text{ "}+k(t)\text{"} \end{aligned}}$$

i.e. you can take any function $\lambda(\vec{r},t)$: $\begin{cases} \text{add } \nabla\lambda \text{ to } \vec{A} \\ \text{add } -\frac{\partial\lambda}{\partial t} \text{ to } V \end{cases}$

$\hookrightarrow$ $\vec{E}$ & $\vec{B}$ remain unchanged

A judicious choice of $\lambda(\vec{r},t)$ can simplify equations (3) and (4)

<u>Coulomb Gauge</u> (also called the "transverse gauge")

we pick λ such that $\boxed{\vec{\nabla}\cdot\vec{A}=0}$ — $\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{\nabla}\lambda = 0 \Rightarrow \underbrace{\nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}}_{\text{"poisson's equation"}}$

↑ in principle one can solve for λ.

↳ In this case (3) → $\vec{\nabla}^2 V + \frac{\partial}{\partial t}\cancel{(\vec{\nabla}\cdot\vec{A})}^{\,0} = -\frac{\rho(\vec{r},t)}{\varepsilon_0}$

$$\Rightarrow \vec{\nabla}^2 V = -\frac{\rho(\vec{r},t)}{\varepsilon_0} \Rightarrow V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}\,d^3r'$$

↳ <u>note:</u> $V(\vec{r},t)$ depends instantaneously on $\rho(\vec{r},t)$.

Q: Are there speed of light issues?

↳ A: <u>No!</u> $\vec{A}$ does not share this instantaneous behavior and
$$\begin{cases} \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla}\times\vec{A} \end{cases}$$

In fact $\vec{A}$ is still relatively hard to compute:

$$(4) \rightarrow \left(\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\cancel{\vec{\nabla}\cdot\vec{A}}^{\,0} + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}(\vec{r},t)$$

$$\Rightarrow \underbrace{\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}}_{\text{unknown}} = \underbrace{-\mu_0\vec{J}(\vec{r},t) + \mu_0\varepsilon_0\vec{\nabla}\left(\frac{\partial V}{\partial t}\right)}_{\text{known}}$$

↑ known from (3)

↳ wave equation with source term → solve for $\vec{A}$

Lorentz Gauge → Lorentz Invariant "$\vec{A}$ & V"
("Lorenz" Gauge) → will be used for the rest of course
→ most widely used gauge.

we pick λ such that $\boxed{\vec{\nabla}\cdot\vec{A} = -\mu_0\varepsilon_0 \frac{\partial V}{\partial t}}$

Given some $\vec{A}$, V then find $\vec{A}'$, V' (i.e. λ) that satisfy the Lorentz gauge.

$$\vec{\nabla}\cdot\vec{A}' = -\mu_0\varepsilon_0 \frac{\partial V'}{\partial t} = \nabla\cdot(\vec{A}+\vec{\nabla}\lambda) = \vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{\nabla}\lambda$$

$$\Rightarrow \nabla^2\lambda = -\vec{\nabla}\cdot\vec{A} - \mu_0\varepsilon_0 \frac{\partial V'}{\partial t}$$

$V - \frac{\partial \lambda}{\partial t}$

$\frac{\partial V}{\partial t} - \frac{\partial^2 \lambda}{\partial t^2}$

$$\Rightarrow \underbrace{\nabla^2\lambda - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\lambda = \underbrace{-\vec{\nabla}\cdot\vec{A} - \frac{1}{c^2}\frac{\partial V}{\partial t}}_{\text{known}}}_{\text{Solvable ... but not easy}}$$

(wave equation with source term)

$$(3) \rightarrow \vec{\nabla}^2 V + \frac{\partial}{\partial t}\underbrace{(\vec{\nabla}\cdot\vec{A})}_{-\mu_0\varepsilon_0\frac{\partial V}{\partial t}} = -\frac{\rho(\vec{r},t)}{\varepsilon_0}$$

$$\Rightarrow \boxed{\nabla^2 V - \mu_0\varepsilon_0\frac{\partial^2 V}{\partial t^2} = \frac{-\rho(\vec{r},t)}{\varepsilon_0}} \quad (3b)$$

$$(4) \longrightarrow \left(\vec{\nabla}^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \vec{\nabla} \cancelto{0}{\left(\underbrace{\vec{\nabla} \cdot \vec{A}}_{-\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} \right)} = -\mu_0 \vec{J}(\vec{r}, t)$$

$$\Rightarrow \boxed{\vec{\nabla}^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}(\vec{r}, t)} \qquad (4L)$$

Define the <u>d'Alembertian</u> : $\Box^2 \equiv \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}$

$$\equiv \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

So Maxwell's equations in Potential formulation become (in Lorentz gauge)

$$\boxed{\begin{aligned} \Box^2 V &= -\frac{\rho(\vec{r}, t)}{\varepsilon_0} \quad (3L) \\ \Box^2 \vec{A} &= -\mu_0 \vec{J}(\vec{r}, t) \quad (4L) \end{aligned}}$$

4 equations & 4 unknowns

Much less intimidating.

↳ wave equations with source terms.