Thursday, November 9, 2017

#1

Retarded Potentials

We want to solve the Lorentz Gauge $\left(\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}\right)$ potential equations

note the homogeneous solutions: i.e. plane wave can be added to any solution.

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) V(\vec{r},t) = -\frac{1}{\epsilon_0}\rho(\vec{r},t) \qquad (1)$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \vec{A}(\vec{r},t) = -\mu_0 \vec{J}(\vec{r},t) \qquad (2)$$

→ we will <u>guess</u> the solutions of these equations.

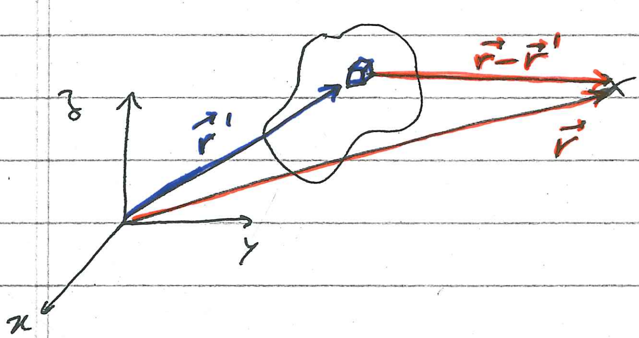
<u>motivation</u>: In the case of electrostatics and magnetostatics (1) & (2) reduce to:

$$\nabla^2 V(\vec{r}) = -\frac{1}{\epsilon_0}\rho(\vec{r})$$

$$\nabla^2 \vec{A}(\vec{r}) = -\mu_0 \vec{J}(\vec{r})$$

and the solutions are $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,d^3r'}{|\vec{r}-\vec{r}\,'|}$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{J(\vec{r}\,')\,d^3r'}{|\vec{r}-\vec{r}\,'|}$$



Suppose that at $t=0$, I make an instantaneous change to the charge density at $\vec{r}\,'$

$$\rho(\vec{r}\,',t) = \begin{cases} \rho_i(\vec{r}\,') & \text{for } t<0 \\ \rho_f(\vec{r}\,') & \text{for } t>0 \end{cases}$$

Q: How quickly does "news" of the change in $\rho(\vec{r}')$ travel to the point $\vec{r}$?

A: At the speed of light c!

We would expect that $V(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ remain unchanged until $t = \frac{|\vec{r}-\vec{r}'|}{c}$.

In otherwords, the potentials at point $\vec{r}$ at time t always experiences the potential produced by $\rho(\vec{r}')$ and $\vec{J}(\vec{r}')$ at a **retarded time**:

$$\boxed{t_{r'} = t - \frac{|\vec{r}-\vec{r}'|}{c}}$$

at $\vec{r}'$ at $\vec{r}$

ex: If point $\vec{r}$ is 3×10^8 m away from $\vec{r}'$, then the potentials at $\vec{r}$ will always be 1s behind ρ & $\vec{J}$ changes at point $\vec{r}'$.

So we expect: $$V(\vec{r},t) = \begin{cases} V(\vec{r};\rho_i) & \text{for } t < \frac{|\vec{r}-\vec{r}'|}{c} \\ V(\vec{r};\rho_f) & \text{for } t \gg \frac{|\vec{r}-\vec{r}'|}{c} \end{cases}$$

maybe there is a "settling time".

Ansatz: there is no settling time.

it turns out that this guess is correct

$$\Rightarrow \boxed{\begin{aligned} V(\vec{r},t) &= \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \frac{\rho(\vec{r}\,', t_{r'})\, d^3r'}{|\vec{r}-\vec{r}\,'|} && (1s) \\ \vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\int_{\text{all space}} \frac{\vec{J}(\vec{r}\,', t_{r'})\, d^3r'}{|\vec{r}-\vec{r}\,'|} && (2s) \end{aligned}}$$

(only for Lorentz Gauge)

Note #1: The above reasoning is not a proof. It can (but yields correct equations) also be applied to $\vec{E}$ & $\vec{B}$ (Coulomb's law & Biot-Savart law) but produces an incorrect solution (i.e. there is a settling time for $\vec{E}$ & $\vec{B}$)

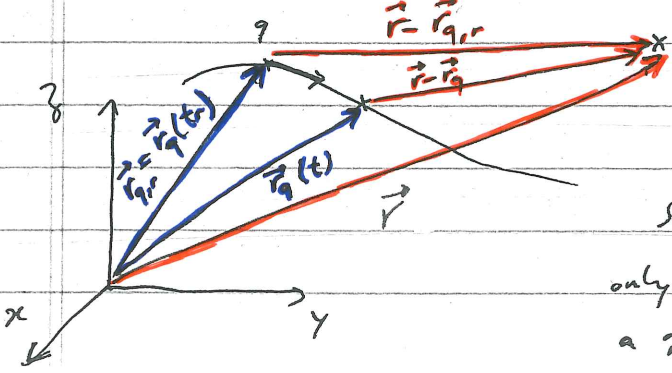
↳ see Griffiths section 10.2.2 p 449–450 (4th Ed.) (Jefimenko's Equations)

Note #2: Replacing $t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c}$ (retarded time) with $t_a = t + \frac{|\vec{r}-\vec{r}\,'|}{c}$ (advanced time) also yields a valid solution. However, it violates causality, so we only consider the retarded time solutions!!!

Retarded Potential of a point charge in motion:

$\vec{r}_q \equiv \vec{r}\,'$ → $\vec{r}_q(t_r)$

Lienard–Wiechert Potentials



$$t_r = t - \frac{|\vec{r}-\vec{r}_q|}{c}$$

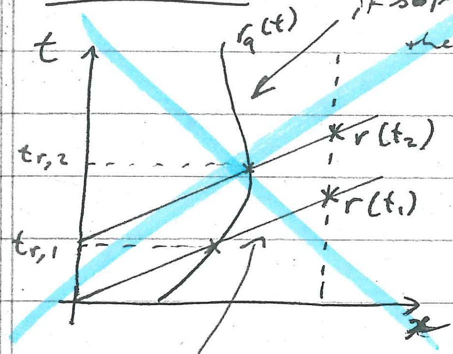
Since q cannot travel faster than light only one position $\vec{r}_q(t_r)$ affects $\vec{r}$ at a given time t.

At each time t, we must figure out t_r.

$$(3) \quad \boxed{|\vec{r} - \vec{r}_q(t_r)| = c\,(t - t_r)} \quad \text{solve for } t_r$$

1D model

if slope on r_q were shallower than $1/c$ then 2 solutions possible (i.e. particle travels faster than c



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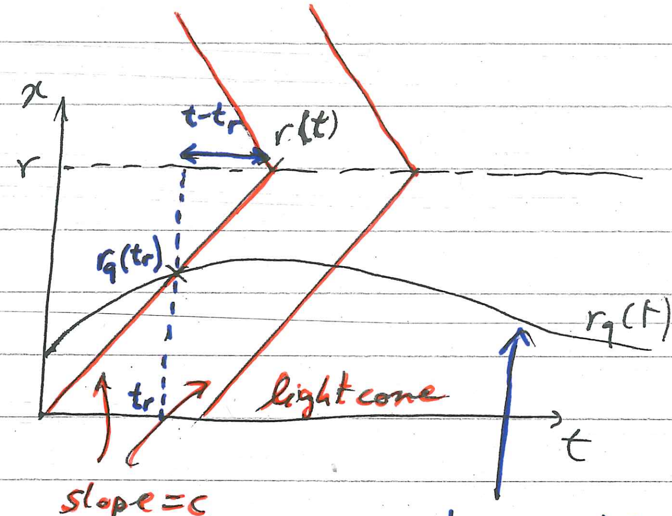
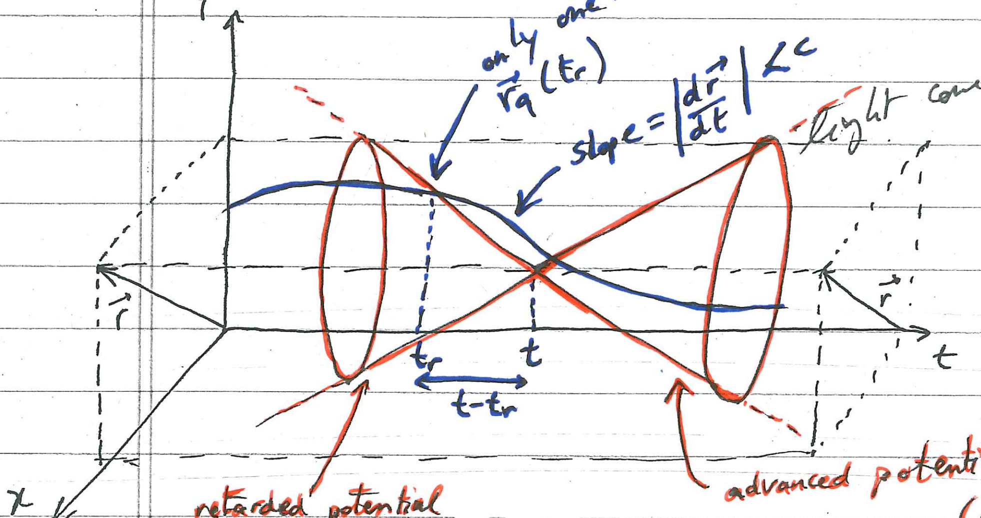


slope cannot be steeper than c otherwise 2 or more solutions possible

2D model



retarded potential light cone (causal)

advanced potential light cone (not casaul)

<u>1st attempt at a solution:</u>

we must solve for $t_{r'}$ for our point charge

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}', t_{r'})\, d^3r'}{|\vec{r}-\vec{r}'|}$$

$$\rho = q\,\delta(\vec{r}' - \vec{r}_q(t_{r'}))$$

$$\cancel{= \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{r}_q(t_r)|}}$$

we have to solve for the correct t_r using equation (3)

ignores the fact that $\delta(\vec{r}' - r_q(t_{r'}))$ is the point limit of a finite distribution

↳ we must evaluate $\int \frac{\rho(\vec{r}', t_{r'})\, d^3r'}{|\vec{r}-\vec{r}'|}$, but each point in the distribution will have a different "$t_{r'}$"

$d^3r'(t_{r'})$

⚠ It is not obvious that this will cause problems for a point charge "distribution".

Effect has nothing to do with ~~length contraction~~ or ~~time dilation~~ (Special Relativity)!

1D model

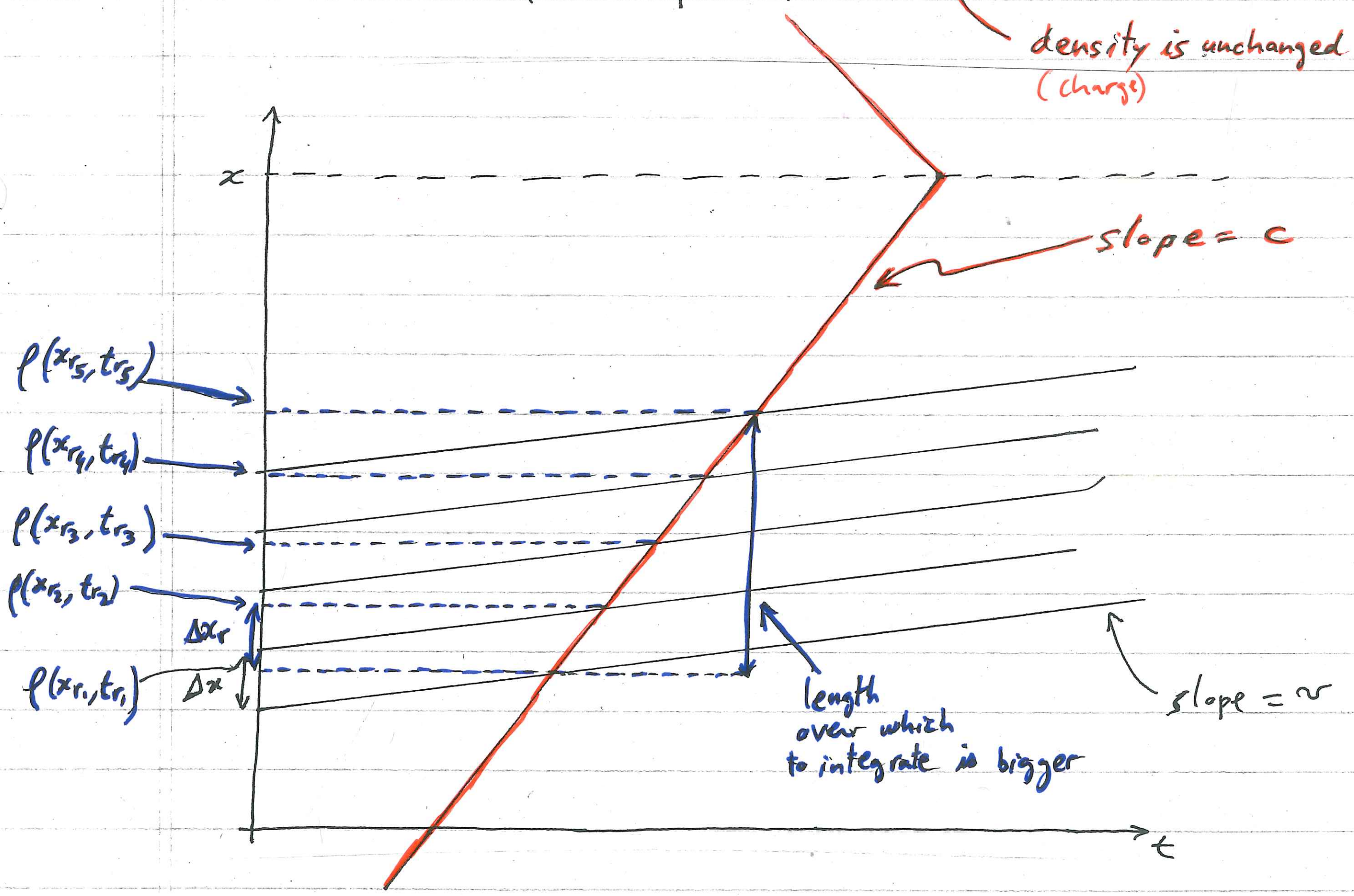
(see also Feynman lectures II-21-10)

$$V(x,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(x_q', t_r')\,dx_q'(t_r')}{|x - x_q'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{|x - \langle x_q(t_r)\rangle|}\int \rho(x_q', t_r')\,dx_q'(t_r')$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{|x - \langle x_q(t_r)\rangle|}\sum_i \rho(x_{q_i}'(t_{r_i}'))\,dx_{q_i}(t_{r_i}')$$

density is unchanged (charge)

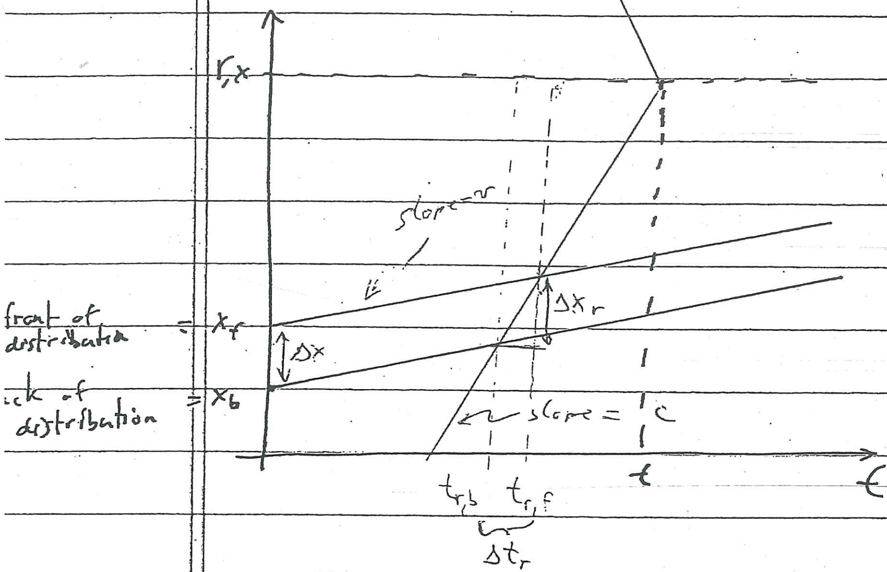


$\Delta x_r > \Delta x$

Some algebra shows that $\Delta x_r = \dfrac{\Delta x}{1 - v/c}$ but at each $x_{q_i}'(t_{r_i}')$ $\rho(x_{q_i}'(t_{r_i}'))$ is the same as in the unretarded case.

Back-up notes

1D model: ⚠ this has nothing to do with ~~length contraction~~ or ~~time dilation~~.
(constant velocity q)



consider a uniform ~~square~~ distribution of charge

$$\rho(x) = \begin{cases} \dfrac{q}{x_f - x_b} & x_b < x < x_f \\ 0 & \text{otherwise} \end{cases}$$

~~for vector~~ with retarded times

~~$\Delta x_r \geq \Delta x$~~

$$\rho(x) = \begin{cases} \dfrac{q}{x_f(t_{r,f}) - x_b(t_{r,b})}, & x_b(t_{r,b}) < x < x_f(t_{r,f}) \\ 0 & \text{otherwise} \end{cases}$$

what's the relationship between Δx & Δx_r?

$x - x_f(t_{r,f}) = c(t - t_{r,f})$

$\Leftrightarrow x - (x_f + v t_{r,f}) = ct - c t_{r,f}$ (6f) $\Rightarrow$

$$\rho(x) = \begin{cases} \dfrac{q}{\Delta x_r} & \text{for } x_b(t_{r,b}) < x < x_f(t_{r,f}) \\ 0 & \text{otherwise} \end{cases}$$

similarly $x - (x_b + v t_{r,b}) = ct - c t_{r,b}$ (6b)

(6f) $-$ (6b) $\Rightarrow$ $-x_f - v(t_{r,f}) + x_b + v t_{r,b} = -c t_{r,f} + c t_{r,b}$

$\Leftrightarrow$ $c(\underbrace{t_{r,f} - t_{r,b}}_{\Delta t_r}) = (\underbrace{x_f - x_b}_{\Delta x}) + v(\underbrace{t_{r,f} - t_{r,b}}_{\Delta t_r})$

$\Leftrightarrow$ $c\,\Delta t_r = \Delta x + v\,\Delta t_r$

$\Leftrightarrow$ $\Delta t_r = \dfrac{\Delta x}{c - v}$

note that $\Delta x_r = c\,\Delta t_r \Rightarrow \boxed{\Delta x_r = \dfrac{c}{c-v}\Delta x} = \dfrac{\Delta x}{\ldots}$

the retarded integral has the same number of Riemann points, but the length of each $dx'_{r,i}$ increases by $\frac{1}{1-\frac{v}{c}} = \frac{\Delta x_r}{\Delta x}$

Thus $\int \rho(x_q'(t_r'), t_r')\, dx_q'(t_r') = \frac{1}{1-\frac{v}{c}} \int \rho(x', t_r')\, dx' = \frac{q}{1-\frac{v}{c}}$

In 3D, we get: (perpendicular components to $\vec{v}$ are unaffected)

$$\boxed{\int \rho(r_q'(t_r'), t_r')\, dr'(t_r') = \frac{q}{1-\frac{\vec{v}\cdot(\widehat{\vec{r}-\vec{r}_q(t_r)})}{c}}}$$

$\vec{v}$ = velocity of charge q at t_r

Stopped here

Thus, $V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}'(t_r'), t_r')\, dr'(t_r')}{|\vec{r}-\vec{r}(t_r')|}$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q}{1-\frac{\vec{v}\cdot\widehat{\vec{r}-\vec{r}_q(t_r)}}{c}} \frac{1}{|\vec{r}-\vec{r}_q(t_r)|}$$

$$\Rightarrow \boxed{V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0} \frac{qc}{|\vec{r}-\vec{r}_q(t_r)|c - \vec{v}\cdot(\vec{r}-\vec{r}_q(t_r))}}$$

at t_r

Similarly,

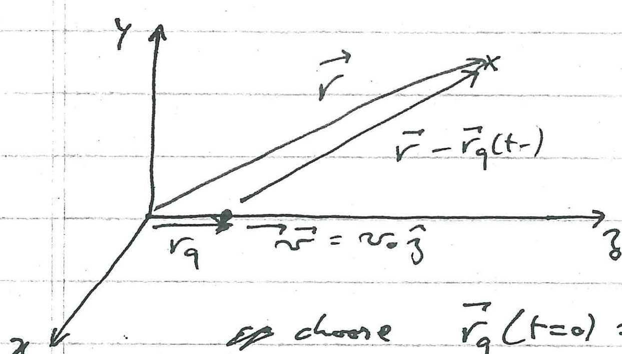
$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{c\,q\,\overbrace{\vec{v}(t_r)}^{\text{"}\vec{J}\text{"}}}{|\vec{r}-\vec{r}_q(t_r)|c - \vec{v}\cdot(\vec{r}-\vec{r}_q(t_r))} = \frac{\vec{v}(t_r)\,V(\vec{r},t)}{c^2}}$$

Lienard – Wiechert electrodynamic potentials for a point charge in ~~uniform~~ motion.

Electromagnetic Fields of moving and accelerating point charges

Point charge in uniform motion

Consider a point charge in uniform motion:



$$t_r = t - \frac{|\vec{r}-\vec{r}_q(t_r)|}{c}$$

choose $\vec{r}_q(t=0) = 0$

$$\vec{r}_q(t) = v_0 t\,\hat{z}$$

$$\vec{r}_q(t_r) = v_0 t_r\,\hat{z}$$

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{|\vec{r}-\vec{r}_q(t_r)|c - \vec{v}(t_r)\cdot(\vec{r}-\vec{r}_q(t_r))}$$

t_r

$$\vec{A}(\vec{r},t) = \frac{\vec{v}(t_r)}{c^2} V(\vec{r},t)$$