Thursday, November 16, 2017

B-field of a point charge in uniform motion

$$\vec{A}(\vec{r},t)=\frac{\vec{v}(t_r)}{c^2}\,V(\vec{r},t)=\frac{q}{4\pi\epsilon_0 c^2}\,v_0\hat{z}\,\frac{1}{\underbrace{\sqrt{1-\left(\frac{v_0}{c}\right)^2}}_{=\gamma}\sqrt{x^2+y^2+\gamma^2\left(\frac{z-v_0t}{\cancel{\sqrt{1-\left(\frac{v_0}{c}\right)^2}}}\right)^2}}$$

Let's calculate $\vec{B}$:

$$\vec{B}=\vec{\nabla}\times\vec{A}$$

$$=\left(\partial_y A_z-\cancelto{0}{\partial_z A_y},\ \cancelto{0}{\partial_z A_x}-\partial_x A_z,\ \cancelto{0}{\partial_x A_y}-\cancelto{0}{\partial_y A_x}\right)$$

$$=\frac{\gamma q}{4\pi\epsilon_0}\frac{v_0}{c^2}\cancel{\left(-\frac{1}{2}\right)}\frac{\left(\cancel{-2}y,\ +\cancel{2}x,\ 0\right)}{\left[x^2+y^2+\gamma^2(z-v_0t)^2\right]^{3/2}}$$

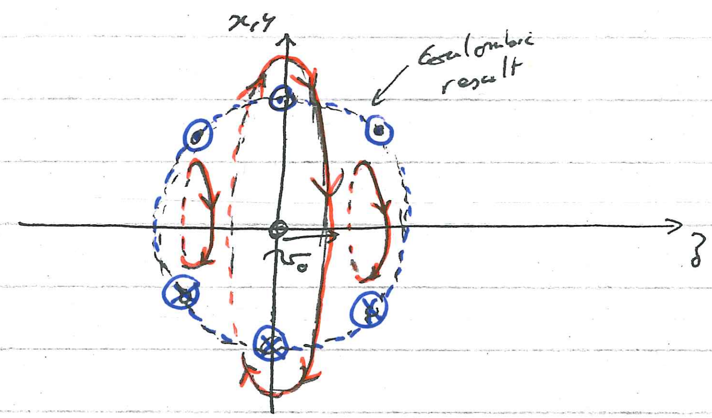
recall: $\vec{E}=\frac{q}{4\pi\epsilon_0}\gamma\frac{(x,\ y,\ z-v_0t)}{\left[x^2+y^2+\gamma^2(z-v_0t)^2\right]^{3/2}}$

$$(0,\ 0,\ 1)\times(x,\ y,\ z)=(-y,\ x,\ 0)$$

$$\Rightarrow\ \boxed{\vec{B}=\frac{\vec{v}_0}{c^2}\times\vec{E}}$$

$\Rightarrow$ $\vec{B}$ is always perpendicular to $\vec{E}$ and $\vec{v}_0$

direction of "current"



if $v \ll c$ so that $\gamma\simeq 1$, then

$$\vec{B}=\frac{\vec{v}\times\hat{r}}{c^2}\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}=\frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$$

Biot-Savart for a "point" current

EM fields of a point charge with arbitrary motion

(i.e. $\frac{d\vec{v}}{dt} \neq 0$)

long derivation (see Griffiths p456-460, 4th ed.)

$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{|\vec{R}|}{(\vec{R}\cdot\vec{u})^3}\Big[\overbrace{\underbrace{(c^2-v^2)\vec{u}}_{\text{at } t_r}}^{\text{velocity term}} + \overbrace{\underbrace{\vec{R}\times(\vec{u}\times\vec{a})}_{\text{at } t_r}}^{\text{acceleration term}}\Big]}$$

$$\vec{E} = \underbrace{\text{"cst"}\,\vec{E}_{near}\cdot\frac{1}{R^2}}_{\text{generalized Coulomb field}} + \underbrace{\text{"cst"}\,\vec{E}_{far}\,\frac{1}{R}}_{\text{depends on acceleration}}$$

↳ dominates at large distances: "radiation field"

define: $\vec{R} = \vec{r} - \vec{r}_q(t_r)$

$\vec{u} = c\hat{R} - \vec{v}(t_r)$

$\vec{a} = \left.\frac{d\vec{v}}{dt}\right|_{t=t_r}$ = acceleration at the retarded time

where did the $\vec{a}$-dependence come from?

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$$

gives rise to $\frac{d\vec{v}}{dt} = \vec{a}$

gives rise to terms with $\frac{\partial\vec{v}}{\partial x} = \underbrace{\frac{d\vec{v}}{dt_r}}_{\vec{a}}\frac{\partial t_r}{\partial x}$

Also, $$\boxed{\vec{B}(\vec{r},t) = \frac{1}{c}\,\hat{R}\times\vec{E}(\vec{r},t)}$$

$$= \overrightarrow{cst\,B_{near}}\cdot\frac{1}{R^2} + \overrightarrow{cst\,B_{far}}\cdot\frac{1}{R}$$

<u>Power Flow</u> : <u>Poynting vector</u>

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \sim \Big(\underbrace{\overrightarrow{cst\,E_{near}}\times\overrightarrow{cst\,B_{near}}}_{S_4}\Big)\frac{1}{R^4}$$
$$+\Big(\underbrace{\overrightarrow{cst\,E_{near}}\times\overrightarrow{cst\,B_{far}}+\overrightarrow{cst\,E_{far}}\times\overrightarrow{cst\,B_{near}}}_{S_3}\Big)\frac{1}{R^3}$$
$$+\Big(\underbrace{\overrightarrow{cst\,E_{far}}\times\overrightarrow{cst\,E_{far}}}_{S_2}\Big)\frac{1}{R^2}$$

= local power per unit area

Total power at large distances $\sim \lim_{R\to+\infty} \eta 4\pi R^2\left(\cancel{\frac{S_4}{R^4}}+\cancel{\frac{S_3}{R^3}}+\frac{S_2}{R^2}\right)$

$$= \eta 4\pi S_2$$

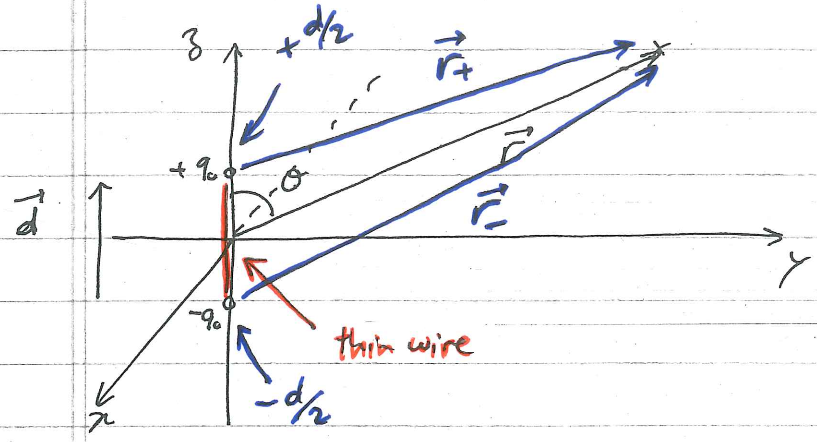
$\Rightarrow$ Only terms due to <u>accelerating charges</u> (or time varying currents) contribute to <u>radiated power</u> far from source.

<u>EM waves are generated by accelerating charges:</u>

1. Accelerating charges always generate EM waves
2. Stationary charges (and currents) do not generate EM waves
charges at constant velocity

Electric Dipole Radiation (Canonical EM wave source i.e. a small antenna)

Consider an electric dipole with charges connected by a thin wire:



- Electric dipole is $\vec{p}_0 = q_0 \vec{d}$

- Consider that the charges oscillate in strength over time.

$$q(t) = q_0 \cos(\omega t) \Rightarrow \vec{p}(t) = q_0 \vec{d} \cos(\omega t) = \vec{p}_0 \cos(\omega t)$$

$$\Rightarrow \text{current: } I(t) = \frac{dq}{dt} = \underbrace{-q_0\omega}_{I_0} \sin(\omega t) = I_0 \sin(\omega t)$$

Objective: Calculate $\vec{E}$ & $\vec{B}$ and $\vec{S}$ for r large.

Roadmap: Calculate V & $\vec{A}$ → make approximations → Calculate $\vec{E}$ & $\vec{B}$ → $\vec{S}$

most of the work is here

Scalar potential

Retarded potential: $$V(\vec r, t) = \frac{1}{4\pi\varepsilon_0} \int_{\text{all space}} \frac{\rho(\vec r\,', t_{r'})}{|\vec r - \vec r\,'|}\, d^3 r'$$

$v = 0$ so there is no "Liénard-Wiechert" δ-function correction

$$= \frac{1}{4\pi\varepsilon_0} \int d^3r' \left\{ q_0 \frac{\delta(\vec r\,' - \vec d/2)\cos(\omega t_{r'})}{|\vec r - \underset{\vec d/2}{\vec r\,'}|} - q_0 \frac{\delta(\vec r\,' + \frac{\vec d}{2})\cos(\omega t_{r'})}{|\vec r - \underset{-\vec d/2}{\vec r\,'}|} \right\}$$

$$= \frac{q_0}{4\pi\varepsilon_0} \left\{ \frac{\cos\left[\omega\left(t - \frac{|\vec r_+|}{c}\right)\right]}{|\vec r_+|} - \frac{\cos\left[\omega\left(t - \frac{|\vec r_-|}{c}\right)\right]}{|\vec r_-|} \right\}$$

$$|\vec r_\pm| = r_\pm = \sqrt{(z \mp d/2)^2 + y^2}$$

$$= \sqrt{\underbrace{z^2 + y^2}_{r^2} \mp \underset{\uparrow\, r\cos\theta}{z}\, d + (d/2)^2}$$

$$= \sqrt{r^2 \mp r d \cos\theta + (d/2)^2}$$ (law of cosines)

Approximation 1: take the limit of a perfect dipole $r \gg d$

$$\hookrightarrow r_\pm = r\sqrt{1 \mp \frac{d}{r}\cos\theta + \left(\frac{d}{2r}\right)^2}$$

$$\simeq r\left(1 \mp \frac{1}{2}\frac{d}{r}\cos\theta + \frac{1}{2}\cancel{\left(\frac{d}{2r}\right)^2}^{\text{neglect}} + \dots\right)$$

$$\Rightarrow \boxed{r_\pm \simeq r\left(1 \mp \frac{d}{2r}\cos\theta\right)}$$

so $$\boxed{\frac{1}{r_\pm} \simeq \frac{1}{r}\,\frac{1}{1 \mp \frac{d}{2r}\cos\theta} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)}$$

we can also simplify the $\cos\left[\omega\left(t-\frac{r_\pm}{c}\right)\right]$ terms:

$$\cos\left[\omega\left(t-\frac{r_\pm}{c}\right)\right] \simeq \cos\left[\omega\left(t-\frac{r}{c}\left(1\mp\frac{d}{2r}\cos\theta\right)\right)\right]$$

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$$\simeq \cos\left[\omega\left(t-\frac{r}{c}\right) \pm \frac{\omega d}{2c}\cos\theta\right]$$

$$\simeq \cos\left[\omega\left(t-r/c\right)\right]\cos\left(\frac{\omega d}{2c}\cos\theta\right) \mp \sin\left[\omega\left(t-r/c\right)\right]\sin\left[\frac{\omega d}{2c}\cos\theta\right]$$

Approximation 2: $\lambda \gg d$

$\lambda f = c \iff \lambda = \frac{c}{f} \iff \frac{\lambda}{2\pi} = \frac{c}{\omega}$

Thus $$\cos\left[\omega\left(t-\frac{r_\pm}{c}\right)\right] = \cos\left[\omega\left(t-\frac{r}{c}\right)\right]\underbrace{\cos\left(\cancel{\frac{\pi d}{\lambda}}\cos\theta\right)}_{\sim 1} \mp \sin\left[\omega\left(t-\frac{r}{c}\right)\right]\underbrace{\sin\left(\cancel{\frac{\pi d}{\lambda}}\cos\theta\right)}_{\sim \frac{\pi d}{\lambda}\cos\theta}$$

(small angle approx.)

$$\simeq \cos\left[\omega\left(t-r/c\right)\right] \mp \frac{\pi d}{\lambda}\cos\theta\,\sin\left[\omega\left(t-r/c\right)\right]$$

back to $V(\vec{r},t)$

Stopped here

$$V(\vec{r},t) \simeq \frac{q_0}{4\pi\varepsilon_0}\frac{1}{r}\left\{\left[\cos\left[\omega\left(t-\frac{r}{c}\right)\right] - \frac{\pi d}{\lambda}\cos\theta\sin\left[\omega\left(t-r/c\right)\right]\right]\left(1+\frac{d}{2r}\cos\theta\right)\right.$$
$$\left. - \left[\cos\left[\omega\left(t-r/c\right)\right] + \frac{\pi d}{\lambda}\cos\theta\sin\left[\omega\left(t-r/c\right)\right]\right]\left(1-\frac{d}{2r}\cos\theta\right)\right\}$$

lots of cancelling

$$\simeq \frac{q_0}{4\pi\varepsilon_0}\frac{1}{r}\left\{\underbrace{\frac{d}{r}\cos\theta\cos\left[\omega\left(t-r/c\right)\right]}_{\substack{\text{quasi-static term}\\ 1/r^2}} - \underbrace{\frac{2\pi d}{\lambda}\cos\theta\sin\left[\omega\left(t-r/c\right)\right]}_{\substack{\text{radiative term}\\ 1/r}}\right\}$$

Approximation #3 : keep only the leading order term ($r \gg \lambda$)
(radiative term)

$$\boxed{\begin{aligned} V(\vec{r},t) &\simeq \frac{-p_0}{4\pi\varepsilon_0}\,\frac{2\pi}{\lambda}\,\frac{\cos\theta}{r}\,\sin\left[\omega\left(t - r/c\right)\right] \\ &\simeq \frac{-p_0}{4\pi\varepsilon_0}\,\frac{\omega}{c}\,\frac{\cos\theta}{r}\,\sin\left[\omega\left(t - r/c\right)\right] \end{aligned}}$$

Vector Potential

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{\text{all space}} \frac{\vec{J}(\vec{r}\,', t_{r'})\, d^3r'}{|\vec{r} - \vec{r}\,'|}$$

limit
$d \ll r$
$d \ll \lambda$
$d \ll \lambda \ll r$

for $-\frac{d}{2} < z < +d/2$

$$\simeq \frac{\mu_0}{4\pi}\int d^3r' \, \underbrace{I_0}_{-q_0\omega}\, \frac{\delta\left(\sqrt{x'^2 + y'^2}\right) \sin\left[\omega\left(t - r/c\right)\right]}{r}\,\hat{z}$$

$$\simeq \frac{\mu_0}{4\pi}\int_{-d/2}^{+d/2} dz' \, (-q_0\omega)\, \frac{\sin\left[\omega\left(t - r/c\right)\right]}{r}$$

$$\Rightarrow \boxed{\vec{A}(\vec{r},t) \simeq \frac{-\mu_0}{4\pi}\, p_0\omega\, \frac{\sin\left[\omega\left(t - r/c\right)\right]}{r}\,\hat{z}}$$

Now we are ready to calculate $\vec{E}$ & $\vec{B}$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \vec{\nabla}\times\vec{A}$$