Tuesday, November 14, 2017

Lienard–Wiechert Potentials for a point charge in motion

Retarded Potential:

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t_{r'})}{|\vec{r}-\vec{r}\,'|}\,d^3r' \quad \text{with } \rho = q\,\delta(\vec{r}\,'-\vec{r}_q(t_{r'}))$$

$$\cancel{= \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}_q(t_r)|}}$$

naïve answer

However, last time we showed that in 1D: $\int q\,\delta(\vec{r}\,'-\vec{r}_q(t_{r'}))\,dr' = \frac{q}{1-v/c}$

in 3D:

$$\boxed{\int \underbrace{\rho(r_q'(t_{r'}),t_{r'})}_{q\,\delta(\vec{r}\,'-\vec{r}_q(t_{r'}))}\,dr'(t_{r'}) = \frac{q}{1-\frac{\vec{v}\cdot(\widehat{\vec{r}-\vec{r}_q(t_r)})}{c}}}$$

$\vec{v}$ = velocity of charge q at t_r

Thus, $$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,'(t_{r'}),t_{r'})\,dr'(t_{r'})}{|\vec{r}-\vec{r}(t_{r'})|}$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{q}{1-\frac{\vec{v}\cdot\widehat{\vec{r}-\vec{r}_q(t_r)}}{c}}\,\frac{1}{|\vec{r}-\vec{r}_q(t_r)|}$$

$$\Rightarrow \boxed{V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\,\frac{qc}{|\vec{r}-\vec{r}_q(t_r)|\,c-\vec{v}\cdot(\vec{r}-\vec{r}_q(t_r))}}$$

at t_r

Similarly,

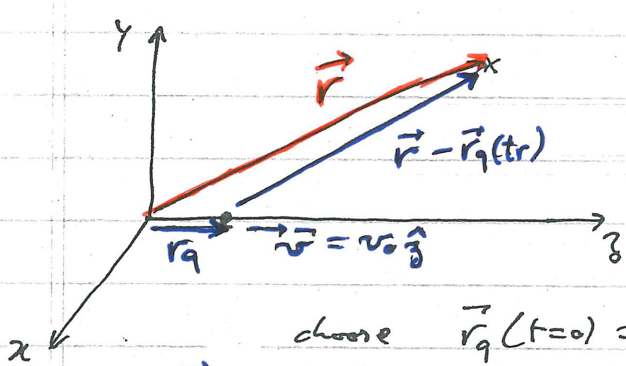
$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{c q \overbrace{\vec{v}(t_r)}^{\text{“}\vec{J}\text{”}}}{|\vec{r}-\vec{r}_q(t_r)|c - \vec{v}\cdot(\vec{r}-\vec{r}_q(t_r))} = \frac{\vec{v}(t_r)\, V(\vec{r},t)}{c^2}}$$

Lienard – Wiechert electrodynamic potentials for a point charge in ~~uniform~~ motion. (in Lorentz gauge)

Electromagnetic Fields of moving and accelerating point charges

Point charge in uniform motion

Consider a point charge in uniform motion:



$$t_r = t - \frac{|\vec{r}-\vec{r}_q(t_r)|}{c}$$

choose $\vec{r}_q(t=0) = 0$

$$\vec{r}_q(t) = v_0 t\,\hat{z}$$

$$\Rightarrow \vec{r}_q(t_r) = v_0 t_r\,\hat{z}$$

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{|\vec{r}-\vec{r}_q(t_r)|c - \vec{v}(t_r)\cdot(\vec{r}-\vec{r}_q(t_r))}$$

t_r

$$\vec{A}(\vec{r},t) = \frac{\vec{v}(t_r)}{c^2} V(\vec{r},t)$$

<u>Roadmap</u>: – Calculate $V(\vec{r},t) \longrightarrow$ get $\vec{A}(\vec{r},t)$

– Calculate $\vec{E}$ & $\vec{B}$ using $\begin{cases} \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$

We need to determine the retarded time $t_r(t)$

retarded time is defined implicitly: $|\vec{r} - \vec{r}_q(t_r)| = c(t - t_r)$

$\vec{r}_q(t_r) = v_0 t_r \hat{z}$

$$\Leftrightarrow \sqrt{x^2 + y^2 + (z - v_0 t_r)^2} = c(t - t_r)$$

$$\Rightarrow x^2 + y^2 + (z - v_0 t_r)^2 = c^2 (t - t_r)^2 \quad (*)$$

$$\Leftrightarrow x^2 + y^2 + z^2 + v_0^2 t_r^2 - 2 z v_0 t_r - c^2 t^2 - c^2 t_r^2 + 2 c^2 t_r t = 0$$

multiply by -1

$$\Leftrightarrow (c^2 - v_0^2) t_r^2 - 2(c^2 t - z v_0) t_r - [(x^2 + y^2 + z^2) - c^2 t^2] = 0$$

$$\Rightarrow t_{r_\pm} = \frac{\cancel{2}(c^2 t - z v_0) \pm \sqrt{\cancel{4}(c^2 t - z v_0)^2 + \cancel{4}(c^2 - v_0^2)(x^2 + y^2 + z^2 - c^2 t^2)}}{\cancel{2}(c^2 - v_0^2)}$$

if $v_0 = 0$ then $$t_{r_\pm} = \frac{c^2 t \pm \sqrt{\cancel{(c^2 t)^2} + c^2(x^2 + y^2 + z^2 - \cancel{c^2 t^2})}}{c^2}$$

$$= t \pm \frac{\sqrt{x^2 + y^2 + z^2}}{c} \quad \begin{cases} + & \text{advanced time} \\ - & \text{retarded time} \end{cases}$$

We will use $t_{r_-} \rightarrow$ retarded time

$$t_r = \frac{(c^2t - z v_0) - \sqrt{(c^2t - z v_0)^2 + (c^2 - v_0^2)(x^2+y^2+z^2-c^2t^2)}}{c^2 - v_0^2}$$

We can now calculate the denominator of $V(\vec{r},t)$:

$$|\vec{r} - \cancelto{v_0 t_r \hat{z}}{\vec{r}_q(t_r)}|\,c - \underbrace{\cancelto{v_0\hat{z}}{\vec{v}(t_r)} \cdot (\vec{r} - \cancelto{v_0 t_r\hat{z}}{\vec{r}_q(t_r)})}_{v_0 z - v_0^2 t_r \;=\; v_0(z - v_0 t_r)}$$

$$= c\sqrt{\underbrace{x^2+y^2+(z-v_0t_r)^2}_{c^2(t-t_r)^2 \quad \text{see}(*)}} - v_0(z - v_0 t_r)$$

$$= c^2(t - t_r) - v_0(z - v_0 t_r)$$

$$= c^2 t - v_0 z - (c^2 - v_0^2)\,t_r \longleftarrow \text{now replace } t_r$$

$$= \cancel{c^2t} - \cancel{v_0 z} - \cancel{(c^2 - v_0^2)}\left\{\frac{(\cancel{c^2t} - \cancel{z v_0}) - \sqrt{(c^2t - zv_0)^2 + (c^2-v_0^2)(x^2+y^2+z^2-c^2t^2)}}{\cancel{c^2 - v_0^2}}\right\}$$

$$= \sqrt{\cancel{c^4t^2} + \cancel{z^2v_0^2} \underline{- 2c^2 t z v_0} + (c^2 - v_0^2)(x^2+y^2) + \underline{c^2z^2} - \cancel{v_0^2 z^2} - \cancel{c^4t^2} + \underline{c^2t^2v_0^2}}$$

$$= \sqrt{c^2(z - v_0 t)^2 + (c^2 - v_0^2)(x^2+y^2)}$$

$$= c\sqrt{(z - v_0t)^2 + \left(1 - \left(\frac{v_0}{c}\right)^2\right)(x^2+y^2)}$$

$$= c\sqrt{1 - \left(\frac{v_0}{c}\right)^2}\;\sqrt{x^2+y^2 + \left(\frac{z - v_0 t}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}}\right)^2}$$

thus

$$\boxed{V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}\;\sqrt{x^2+y^2+\left(\frac{z-v_0t}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}}\right)^2}}}$$

and $$\boxed{\vec{A}(\vec{r},t) = \frac{q v_0 \hat{z}}{4\pi \underbrace{\cancel{\varepsilon_0} \cancel{c^2}}_{1/\mu_0}} \frac{1}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}\sqrt{x^2+y^2+\left(\frac{z-v_0 t}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}}\right)^2}}}$$

$$\frac{\vec{v}(t_r)V(\vec{r},t)}{c^2}$$

first hints of lorentz transformation

$$x \to x$$
$$y \to y$$
$$z \to \frac{z - v_0 t}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}}$$

note: for $v_0 = 0$

$$\begin{cases} V = \frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{x^2+y^2+z^2}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \\ \vec{A} = 0 \quad \Rightarrow \text{ no } B\text{-field} \end{cases}$$

Let's get $\vec{E}$ & $\vec{B}$: $\quad \vec{E} = -\vec{\nabla} V(\vec{r},t) - \frac{\partial \vec{A}}{\partial t}$

$$\vec{\nabla} V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0} \underbrace{\frac{1}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}}}_{\gamma = \frac{1}{\sqrt{1-(v_0/c)^2}}} (\partial_x, \partial_y, \partial_z) \frac{1}{\sqrt{x^2+y^2+\left(\frac{z-v_0 t}{\sqrt{1-(v_0/c)^2}}\right)^2}}$$

$$= \frac{\gamma q}{4\pi\varepsilon_0}\left(-\frac{1}{\cancel{2}}\right) \frac{1}{\left[x^2+y^2+\gamma^2(z-v_0 t)^2\right]^{3/2}} \left(\cancel{2}x, \cancel{2}y, \gamma^2 \cancel{2}(z-v_0 t)\right)$$

$$= \frac{-\gamma q}{4\pi\varepsilon_0} \frac{\left(x, y, \gamma^2(z-v_0 t)\right)}{\left[x^2+y^2+\gamma^2(z-v_0 t)^2\right]^{3/2}}$$

$$\frac{\partial \vec{A}}{\partial t} = \frac{\gamma q}{4\pi\varepsilon_0} \frac{v_0 \hat{z}}{c^2} \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{x^2+y^2+\gamma^2(z-v_0t)^2}}\right]$$

$$= \frac{\gamma q \, v_0}{4\pi\varepsilon_0 c^2}\,\hat{z}\left(\frac{-1}{\cancel{2}}\right)\frac{\gamma^2 \cancel{2}(z-v_0t)(-v_0)}{\left[x^2+y^2+\gamma^2(z-v_0t)^2\right]^{3/2}}$$

$$= \frac{\gamma q}{4\pi\varepsilon_0}\frac{v_0^2}{c^2}\left(0,\ 0,\ \frac{\gamma^2(z-v_0t)}{\left[x^2+y^2+\gamma^2(z-v_0t)^2\right]^{3/2}}\right)$$

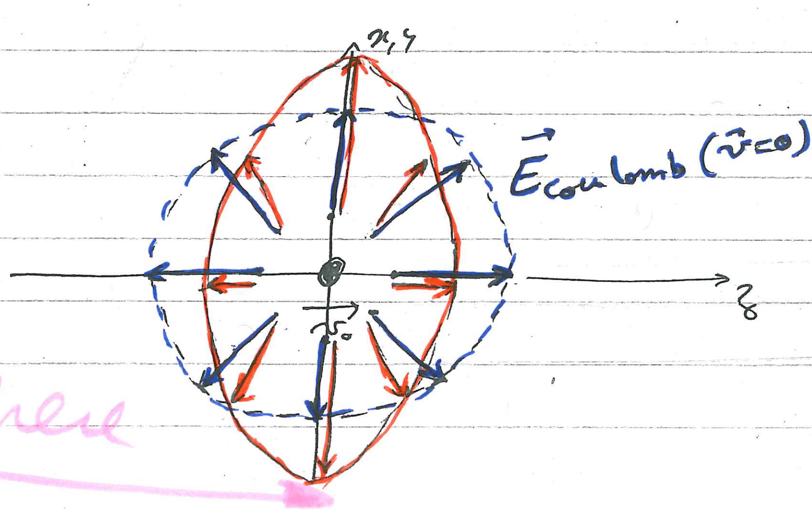
Thus $\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$

$$= \frac{q}{4\pi\varepsilon_0}\gamma\frac{\left(x,\ y,\ \cancel{\gamma^2}(z-v_0t)\cancel{\left(1-\left(\frac{v_0}{c}\right)^2\right)}\right)}{\left[x^2+y^2+\gamma^2(z-v_0t)^2\right]^{3/2}}$$

$$\Longleftrightarrow \boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\gamma\frac{(x,\ y,\ z-v_0t)}{\left[x^2+y^2+\gamma^2(z-v_0t)^2\right]^{3/2}}}$$

if $v_0 = 0$ then $\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$

at $t=0$ with $v \lesssim c$



$\vec{E}(v_0)$ is reduced in the direction/axis of motion, but increased in direction perpendicular to motion

Stopped here

also, $\vec{B} = \vec{\nabla} \times \vec{A}$

$$= \left(\partial_y A_z - \cancel{\partial_z A_y}^{\,0},\ \cancel{\partial_z A_x}^{\,0} - \partial_x A_z,\ \cancel{\partial_x A_y}^{\,0} - \cancel{\partial_y A_x}^{\,0} \right)$$

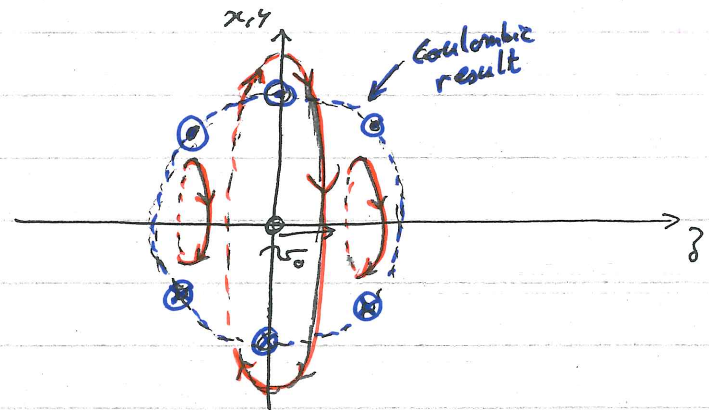
$$= \frac{\gamma q}{4\pi\varepsilon_0} \frac{v_0}{c^2} \cancel{\left(-\frac{1}{2}\right)} \frac{\left(\cancel{-2}y,\ +\cancel{2}x,\ 0\right)}{\left[x^2 + y^2 + \gamma^2 (z - v_0 t)^2\right]^{3/2}}$$

$$\begin{pmatrix} 0, & 0, & 1 \\ x, & y, & z \end{pmatrix}_{\times} = (-y, x, 0)$$

$$\Rightarrow \boxed{\vec{B} = \frac{\vec{v}_0}{c^2} \times \vec{E}}$$

$\Rightarrow$ $\vec{B}$ is always perpendicular to $\vec{E}$ and $\vec{v}_0$

direction of "current"



if $v \ll c$ so that $\gamma \simeq 1$, then

$$\vec{B} = \frac{\vec{v} \times \hat{r}}{c^2} \frac{q}{4\pi\varepsilon_0} \frac{1}{r^2} = \frac{\mu_0}{4\pi} \frac{q \vec{v} \times \hat{r}}{r^2}$$

Biot-Savart for a "point" current