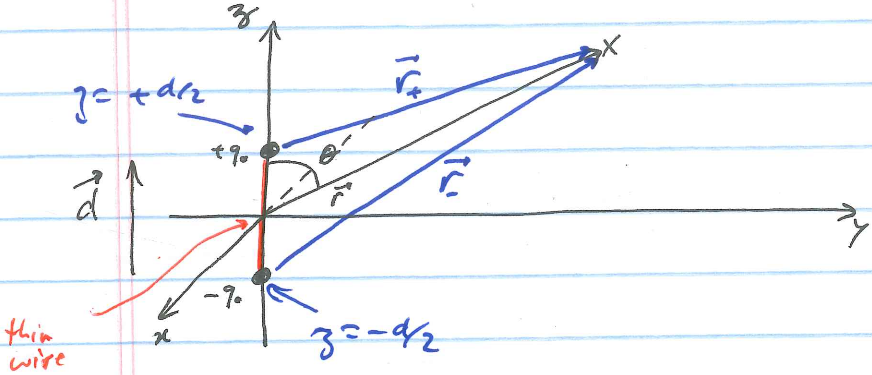
Tuesday, November 21, 2017

Electric Dipole Radiation (continued)



$p_0 = q_0 d$

$q(t) = q_0 \cos(\omega t)$

$\Rightarrow \vec{p}(t) = \vec{p}_0 \cos(\omega t)$

current: $I(t) = \frac{dq}{dt} = \underbrace{-q_0\omega}_{}\sin(\omega t) = \underline{\underline{I_0}} \sin(\omega t)$

Scalar potential:

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0} \int_{\text{all space}} \frac{\rho(\vec{r}\,', t_r')}{|\vec{r}-\vec{r}\,'|} d^3r'$$

$$\vdots$$

$$= \frac{q_0}{4\pi\varepsilon_0} \left\{ \frac{\cos\left[\omega\left(t - \frac{|\vec{r}_+|}{c}\right)\right]}{|\vec{r}_+|} - \frac{\cos\left[\omega\left(t - \frac{|\vec{r}_-|}{c}\right)\right]}{|\vec{r}_-|} \right\}$$

Approximation #1: $r \gg d \Rightarrow \frac{1}{r_\pm} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$

Approximation #2: $\lambda \gg d$

$$\cos\left[\omega\left(t - \frac{r_\pm}{c}\right)\right] \simeq \cos\left[\omega\left(t - r/c\right)\right] \mp \frac{\pi d}{\lambda}\cos\theta \sin\left[\omega\left(t - r/c\right)\right]$$

Stopped here last time

Plug into $V(\vec{r},t)$

$$V(\vec{r},t) \simeq \frac{q_0}{4\pi\varepsilon_0}\frac{1}{r}\left\{\left[\cos\left[\omega\left(t - r/c\right)\right] - \frac{\pi d}{\lambda}\cos\theta \sin\left[\omega\left(t-r/c\right)\right]\right]\left(1 + \frac{d}{2r}\cos\theta\right)\right.$$

lots of cancelling

$$\left. - \left[\cos\left[\omega\left(t - r/c\right)\right] + \frac{\pi d}{\lambda}\cos\theta\sin\left[\omega\left(t - r/c\right)\right]\right]\left(1 - \frac{d}{2r}\cos\theta\right)\right\}$$

$$\simeq \frac{q_0}{4\pi\varepsilon_0}\frac{1}{r}\left\{\underbrace{\frac{d}{r}\cos\theta\cos\left[\omega\left(t - r/c\right)\right]}_{\text{quasistatic term } 1/r^2} - \underbrace{\frac{2\pi d}{\lambda}\cos\theta\sin\left[\omega\left(t - r/c\right)\right]}_{\text{radiative term } 1/r}\right\}$$

Approximation #3: keep only the leading order term $(r \gg \lambda)$

(radiative term)

$$\boxed{\begin{aligned} V(\vec{r},t) &\simeq \frac{-p_0}{4\pi\varepsilon_0}\,\frac{2\pi}{\lambda}\,\frac{\cos\theta}{r}\,\sin[\omega(t-r/c)] \\ &\simeq \frac{-p_0}{4\pi\varepsilon_0}\,\frac{\omega}{c}\,\frac{\cos\theta}{r}\,\sin[\omega(t-r/c)] \end{aligned}}$$

Vector Potential

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{\text{all space}} \frac{\vec{J}(\vec{r}\,', t_{r'})\, d^3r'}{|\vec{r}-\vec{r}\,'|}$$

limit
$d \ll r$
$d \ll \lambda$
$d \ll \lambda \ll r$

for $-\frac{d}{2} < z < +\frac{d}{2}$

$$\simeq \frac{\mu_0}{4\pi}\int d^3r' \,\frac{I_0\,\delta(\sqrt{x'^2+y'^2})\,\sin[\omega(t-r/c)]}{r}\,\hat{z}$$

$I_0 = -q_0\omega$

$$\simeq \frac{\mu_0}{4\pi}\int_{-d/2}^{+d/2} dz' \,(-q_0\omega)\,\frac{\sin[\omega(t-r/c)]}{r}$$

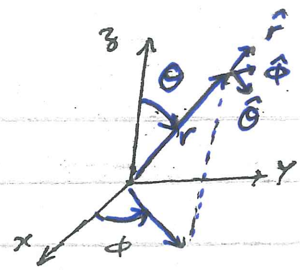
$\frac{d}{2} - (-\frac{d}{2}) = d$

$$\Rightarrow \boxed{\vec{A}(\vec{r},t) \simeq -\frac{\mu_0}{4\pi}\,p_0\omega\,\frac{\sin[\omega(t-r/c)]}{r}\,\hat{z}}$$

Now we are ready to calculate $\vec{E}$ & $\vec{B}$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

switch to spherical coordinates:



$$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \cancel{\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}} \quad 0$$

$$= -\frac{p_0}{4\pi\varepsilon_0}\frac{\omega}{c}\left\{\left[\cancel{-\frac{\sin[\omega(t-r/c)]}{r^2}} - \frac{\omega}{c}\frac{\cos[\omega(t-r/c)]}{r}\right]\cos\theta\,\hat{r}\right.$$

$$\left. - \sin\theta\,\cancel{\frac{\sin[\omega(t-r/c)]}{r^2}}\,\hat{\theta}\right\}$$

neglect higher order terms in $\frac{1}{r}$

$$\Rightarrow \vec{\nabla} V \simeq \frac{p_0}{4\pi\varepsilon_0}\frac{\omega^2}{c^2}\frac{\cos\theta}{r}\cos[\omega(t-r/c)]\,\hat{r}$$

$$\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0}{4\pi}p_0\omega^2\frac{\cos[\omega(t-r/c)]}{r}\,\hat{z} \rightarrow \hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

$$= -\frac{p_0}{4\pi\varepsilon_0}\frac{\omega^2}{c^2}\frac{\cos[\omega(t-r/c)]}{r}\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

Thus
$$\boxed{\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = -\frac{p_0}{4\pi\varepsilon_0}\frac{\omega^2}{c^2}\frac{\sin\theta}{r}\cos[\omega(t-r/c)]\,\hat{\theta}}$$

Finally, $\vec{B} = \vec{\nabla} \times \vec{A}$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\cancel{(\sin\theta A_\phi)}^{0} - \cancel{\frac{\partial A_\theta}{\partial\phi}}^{0}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\cancel{\frac{\partial A_r}{\partial\phi}}^{0} - \frac{\partial}{\partial r}\cancel{(rA_\phi)}^{0}\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

$$\Rightarrow \vec{B} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

with $\vec{A} = \frac{-p_0}{4\pi\varepsilon_0}\frac{\omega}{c^2}\frac{\sin[\omega(t-r/c)]}{r}\underbrace{(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})}_{\hat{z}}$

$$\Rightarrow \vec{B} = \frac{-p_0}{4\pi\varepsilon_0}\frac{\omega}{c^2}\frac{1}{r}\left\{+\frac{\omega}{c}\sin\theta\cos[\omega(t-r/c)] + \cancel{\frac{\sin\theta}{r}\sin[\omega(t-r/c)]}\right\}\hat{\phi}$$

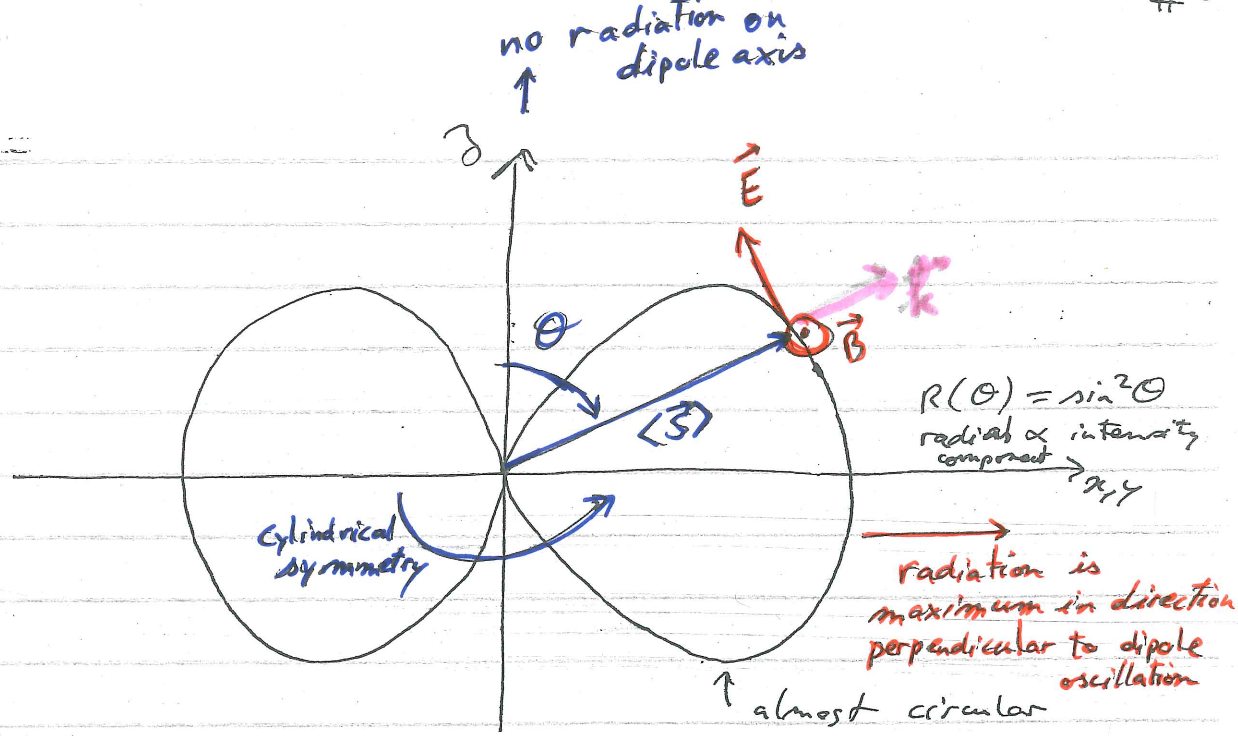
neglect higher order terms in $1/r$

Thus
$$\boxed{\vec{B} = \frac{-p_0}{4\pi\varepsilon_0}\frac{\omega^2}{c^3}\frac{\sin\theta}{r}\cos[\omega(t-r/c)]\,\hat{\phi} = \frac{\hat{r}\times\vec{E}}{c}}$$

Poynting Vector

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{1}{\mu_0}\left(\frac{p_0}{4\pi\varepsilon_0}\right)^2\frac{\omega^4}{c^5}\frac{\sin^2\theta}{r^2}\underbrace{\cos^2[\omega(t-r/c)]}_{\langle\cos^2\rangle_t = 1/2}\,\hat{r}$$

$$\boxed{\text{Intensity} = \langle\vec{S}\rangle = \frac{p_0^2}{32\pi^2\varepsilon_0}\frac{\omega^4}{c^3}\frac{\sin^2\theta}{r^2}\hat{r} = \frac{\mu_0 p_0^2}{32\pi^2}\frac{\omega^4}{c}\frac{\sin^2\theta}{r^2}\hat{r}}$$


no radiation on dipole axis
z
$\vec{E}$
$\vec{k}$
$\vec{B}$
θ
$\langle \vec{S} \rangle$
$R(\theta) = \sin^2\theta$
radial component $\propto$ intensity
x, y
cylindrical symmetry
radiation is maximum in direction perpendicular to dipole oscillation
almost circular