Thursday, November 28, 2011

Power radiated by an accelerating point charge

The $\vec{E}$ & $\vec{B}$ fields generated by ~~for~~ a point charge with ~~in~~ arbitrary motion are

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{|\vec{R}|}{|\vec{R}\cdot\vec{u}|^3}\left[(c^2-\underbrace{v^2}_{\vec{v}(t_r)})\vec{u} + \underbrace{\vec{R}\times(\vec{u}\times\overbrace{\vec{a}}^{\vec{a}(t_r)})}_{\text{only this term produces radiation because it is "}\frac{1}{r}\text{"}}\right]$$

$$\vec{B} = \frac{1}{c}\hat{R}\times\vec{E}$$

$$\text{with } \vec{R} = \vec{r} - \vec{r}_q(t_r)$$
$$\vec{u} = c\hat{R} - \vec{v}(t_r)$$

$$\text{so } \vec{E}_{radiation} = \frac{q}{4\pi\epsilon_0}\frac{|\vec{R}|}{|\vec{R}\cdot\vec{u}|^3}\left[\vec{R}\times(\vec{u}\times\vec{a})\right]$$

if the particle is at rest or nearly at rest $v \ll c$, then $\vec{u} \simeq c\hat{R}$ and

$$\vec{E}_{radiation} \simeq \frac{q}{4\pi\epsilon_0}\frac{\cancel{|\vec{R}|}}{|\vec{R}|^3 c^{\cancel{3}2}}\,\cancel{c}\,\cancel{|\vec{R}|}\left[\hat{R}\times(\hat{R}\times\vec{a})\right]$$

$$\simeq \frac{q}{4\pi\epsilon_0 c^2}\frac{1}{|\vec{R}|}\left[\hat{R}\times(\hat{R}\times\vec{a})\right] \Rightarrow \vec{E}_{Rad}\perp\hat{R}$$

$$\vec{S}_{Radiation} = \frac{1}{\mu_0}\left(\vec{E}_{Radiation}\times\vec{B}_{Radiation}\right)$$

$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

#2

$$\Rightarrow \vec{S}_{Radiation} = \frac{1}{\mu_0}\,\vec{E}_{Radiation} \times \left(\left(\frac{1}{c}\right)\hat{R}\times\vec{E}_{radiation}\right)$$

vector identity

$$= \frac{1}{\mu_0 c}\left[\vec{E}^2_{radiation}\hat{R} - \underbrace{(\hat{R}\cdot\vec{E}_{Radiation})}_{=0 \text{ since } \vec{E}_{rad}\perp\vec{R}}\vec{E}_{Radiation}\right]$$

$$\Rightarrow \boxed{\vec{S}_{Radiation} = \frac{1}{\mu_0 c}\,\vec{E}^2_{Radiation}\,\hat{R}}$$

$\varepsilon_0 c$

$$\Rightarrow \vec{S}_{Radiation} = \frac{1}{\cancel{\mu_0} c}\,\frac{q^2}{16\pi^2\varepsilon_0^{\cancel{2}} c^{\cancel{4}2}}\,\frac{1}{R^2}\,\underbrace{\left[\hat{R}\times(\hat{R}\times\vec{a})\right]^2}_{\underbrace{\left[(\hat{R}\cdot\vec{a})\hat{R} - \vec{a}\right]^2}}\,\hat{R}$$

$$(\hat{R}\cdot\vec{a})^2 + \vec{a}^2 - 2(\hat{R}\cdot\vec{a})(\hat{R}\cdot\vec{a})$$

$$= \vec{a}^2 - (\hat{R}\cdot\vec{a})^2$$

$$\text{thus } \vec{S}_{radiation} = \frac{q^2}{16\pi^2\varepsilon_0 c^3}\,\frac{1}{R^2}\left[\underbrace{\vec{a}^2 - (\hat{R}\cdot\vec{a})^2}_{\vec{a}^2\underbrace{(1-\cos^2\theta)}_{\sin^2\theta}}\right]\hat{R}$$

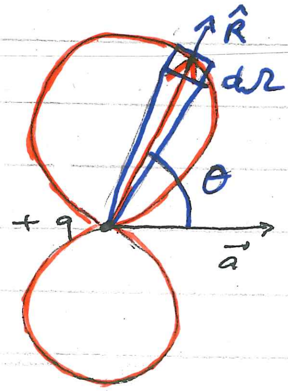
θ = angle between $\vec{a}$ and $\hat{R}$

$$= \frac{\mu_0 q^2}{16\pi^2 c}\,a^2\left(\frac{\sin\theta}{R}\right)^2\hat{R}$$

$\Rightarrow$ dipole-type radiation along acceleration axis

$$dPower = \vec{S}\cdot\hat{n}\,\underbrace{R^2\sin\theta\, d\theta\, d\phi}_{d\Omega}$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2 \sin^2\theta}{16\pi^2 c}$$



__not__ true when $v \approx c$

$\hookrightarrow$ radiation becomes "toa".

$$P_{total} = \oint_{\substack{\text{surface}\\ \text{far away}\\ \text{from source}}} \vec{S}_{radiation} \cdot d\vec{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{1}{R^2} \int \sin^2\theta \, R^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{1}{R^2} R^2 \, 2\pi \underbrace{\int_0^\pi \underbrace{\sin^3\theta}_{\sin\theta - \cos^2\theta \sin\theta} d\theta}_{4/3}$$

$$\Rightarrow \boxed{P_{total} = \frac{\mu_0 q^2 a^2}{6\pi c}}$$

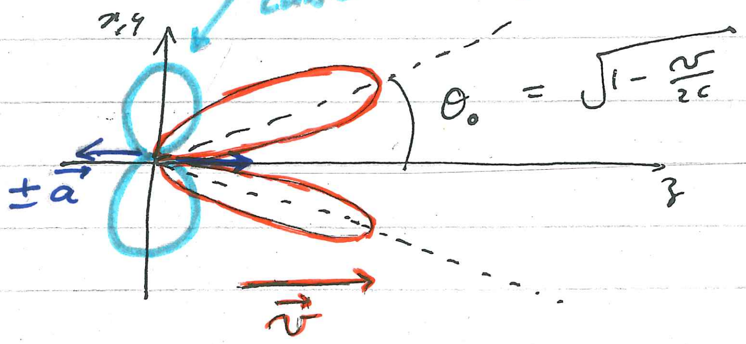
Larmor formula (valid for $v \ll c$)

accelerating charges at relativistic velocities ($v \lesssim c$)

case 1: $\vec{v} \parallel \vec{a}$ e.g. charged particle in a linear accelerator or a high speed charged particle entering a target (Bremsstrahlung)

does not depend on direction $\pm$ of $\underline{a}$.

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2\theta}{\underbrace{\left(1 - \frac{v}{c}\cos\theta\right)^5}_{\text{strong dependence on } v/c}} \qquad d\Omega = \sin\theta \, d\theta \, d\phi$$

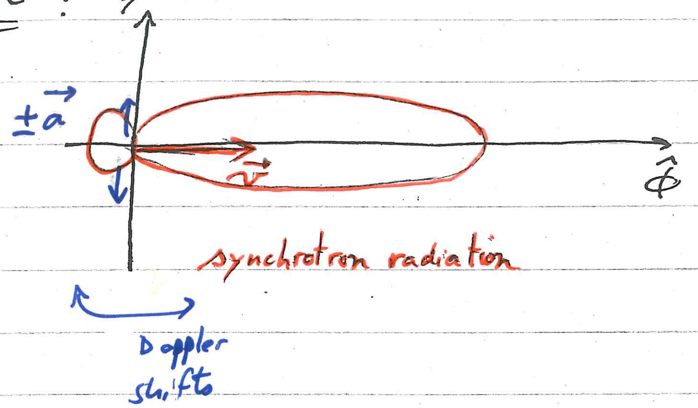


a charged particle in a linear accelerator produces a lot of near collinear radiation (x-rays) (⚠ watch out) a charged, high speed particle entering a material will undergo breaking radiation (Bremsstrahlung)

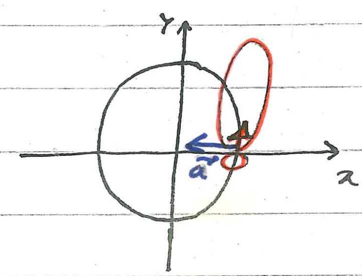
↳ { – accelerated particle incident on a target.
  – cosmic rays entering atmosphere/material.

<u>Case 2:</u> $\vec{a} \perp \vec{v}$ (i.e. synchrotron/cyclotron radiation from a circulating, charged particle.

<u>$v \lesssim c$</u> : "synchrotron radiation"

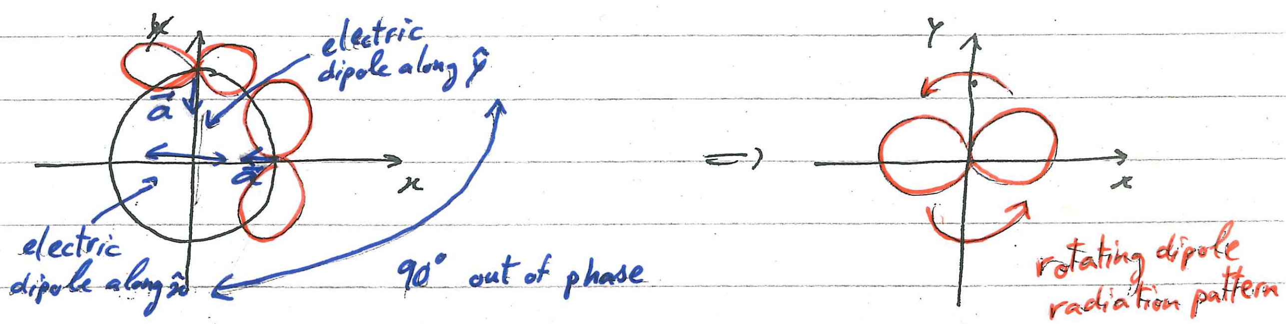


<u>overhead view</u>



"head light" effect for $v \lesssim c$

<u>$v \ll c$</u> : cyclotron radiation



<u>Radiation Reaction</u>

If an accelerating particle radiates energy, then there must be a reaction force associated with this energy loss. (Sort of like a friction force, but for acceleration).
(air resistance)

non-relativistic case ($v \ll c$) [Griffiths 11.2.2]

Larmor's formula: $\text{Power} = \dfrac{d\overset{\text{energy}}{W}}{dt} = \dfrac{\mu_0 q^2 a^2}{6\pi c}$

1st approach: $\vec{F}_{\substack{\text{radiation}\\\text{reaction}}} \cdot \vec{v} \;\cancel{=}\; -\dfrac{\mu_0 q^2 a^2}{6\pi c}$

recoil force on charge due to its EM fields

this assumes that all the energy goes into the radiation field $\Rightarrow$ not true, since a moving charge generates a magnetic field.

↳ these velocity-modified near-fields contain energy from the motion, and are not easy to calculate.

2nd approach: consider a system with periodic motion, so that $\vec{v}(t_i) = \vec{v}(t_f)$

$\vec{a}(t_i) = \vec{a}(t_f)$

t_i & t_f are the time at the start and end of a period.

In this case any energy put into the near-fields is removed after one period; thus we can write

$$\Delta E_{\substack{\text{removed}\\\text{in one}\\\text{period}}} = \int_{t_i}^{t_f} \vec{F}_{\substack{\text{radiation}\\\text{reaction}}} \cdot \vec{v}\, dt = -\frac{\mu_0 q^2}{6\pi c} \int_{t_i}^{t_f} \vec{a}^{\,2}\, dt$$

$$= -\frac{\mu_0 q^2}{6\pi c}\int_{t_i}^{t_f}\left(\frac{d\vec{v}}{dt}\right)\cdot\left(\frac{d\vec{v}}{dt}\right)dt$$

integration by parts

"u" $= \frac{d\vec{v}}{dt} \Rightarrow$ "du" $= \frac{d^2\vec{v}}{dt^2}$

"dv" $= \left(\frac{d\vec{v}}{dt}\right)dt \Rightarrow$ "v" $= \vec{v}$

$$\Leftrightarrow \int_{t_i}^{t_f}\vec{F}_{\substack{\text{radiation}\\ \text{reaction}}}\cdot\vec{v}\,dt = -\frac{\mu_0 q^2}{6\pi c}\left[\underbrace{\cancel{\vec{v}\cdot\frac{d\vec{v}}{dt}\Big|_{t_i}^{t_f}}}_{\Rightarrow 0 \text{ by construction}} - \int_{t_i}^{t_f}\underbrace{\left(\frac{d^2\vec{v}}{dt^2}\right)}_{\frac{d\vec{a}}{dt}}\cdot\vec{v}\,dt\right]$$

$\vec{v}(t_i) = \vec{v}(t_f)$

$\vec{a}(t_i) = \vec{a}(t_f)$

$$\Leftrightarrow \int_{t_i}^{t_f}\left(\vec{F}_{\substack{\text{radiation}\\ \text{reaction}}} - \frac{\mu_0 q^2}{6\pi c}\frac{d\vec{a}}{dt}\right)\cdot\vec{v}\,dt = 0$$

a possible solution is $\boxed{\vec{F}_{\substack{\text{radiation}\\ \text{reaction}}} = \frac{\mu_0 q^2}{6\pi c}\frac{d\vec{a}}{dt}}$

for component $\parallel$ to $\vec{v}$

(Abraham-Lorentz formula)

This is the force of the particle on itself

Stopped here

The formula is problematic! $\Rightarrow$ Big outstanding problem in classical EM

equation of motion $\quad m a = \frac{\mu_0 q^2}{6\pi c}\frac{da}{dt}$

$\boxed{\text{if } \frac{d\vec{a}}{dt} = 0 \Rightarrow \text{suggests that } \vec{F}_{\substack{\text{radiation}\\ \text{reaction}}} = 0 \Rightarrow \text{false}}$

↳ solution $\quad a(t) = a_0 e^{t/\tau}$

with $\tau = \frac{\mu_0 q^2}{6\pi c m}$

$\Rightarrow$ runaway acceleration $\Rightarrow$ contradicts periodic motion hypothe

⚠ avoid formula unless you have no choice!!!

↳ use conservation of energy to get overall effect of radiation reaction