Thursday, November 30, 2017

Radiation Reaction (continued)

Abraham-Lorentz formula : $\boxed{\vec{F}_{radiation} = \frac{\mu_0 q^2}{6\pi c} \frac{d\vec{a}}{dt}}$

This is the force of the charge's EM field on the charge

The formula is problematic! $\Rightarrow$ big outstanding problem in classical EM

equation of motion : $ma = \frac{\mu_0 q^2}{6\pi c} \frac{da}{dt}$

$\hookrightarrow$ solution : $a(t) = a_0 e^{t/\tau}$

with $\tau = \frac{\mu_0 q^2}{6\pi c m}$

$\Rightarrow$ runaway acceleration

$\hookrightarrow$ contradicts periodic motion hypothesis

if $\frac{d\vec{a}}{dt} = 0$ ~~⇒~~

then $\vec{F}_{radiation} = 0$

$\hookrightarrow$ "false"

⚠ avoid formula unless you have no choice!!!

$\hookrightarrow$ use conservation of energy (or momentum) to get the overall effect of radiation reaction.

Lorentz Transformation, Special Relativity, and 4-vectors

Lorentz Transformation

We saw previously that for a moving point charge, the associated scalar & vector potentials, and the associated $\vec{E}$ & $\vec{B}$ fields do not transform according to Gallilean relativity

Stationary Point Charge (Lorentz gauge)		Point charge in uniform motion ($\vec{v}_0 = v_0\hat{z}$)
$V(\vec{r},t) = \dfrac{q}{4\pi\varepsilon_0}\dfrac{1}{\lvert\vec{r}-\vec{r}_0\rvert}$	$\longrightarrow$	$V(\vec{r},t) = \dfrac{q}{4\pi\varepsilon_0}\,\gamma\,\dfrac{1}{\sqrt{x^2+y^2+[\gamma(z-v_0t)]^2}}$
$\vec{A}(\vec{r},t) = 0$	$\longrightarrow$	$\vec{A}(\vec{r},t) = \dfrac{\mu_0}{4\pi}\,qv_0\hat{z}\,\gamma\,\dfrac{1}{\sqrt{x^2+y^2+[\gamma(z-v_0t)]^2}}$
$\vec{E}(\vec{r},t) = \dfrac{q}{4\pi\varepsilon_0}\dfrac{1}{\lvert\vec{r}-\vec{r}_0\rvert^2}$	$\longrightarrow$	$\vec{E}(\vec{r},t) = \dfrac{q}{4\pi\varepsilon_0}\,\gamma\,\dfrac{(x,y,z-v_0t)}{\left[x^2+y^2+[\gamma(z-v_0t)]^2\right]^{3/2}}$
$\vec{B}(\vec{r},t) = 0$	$\longrightarrow$	$\vec{B}(\vec{r},t) = \dfrac{\vec{v}_0\times\vec{E}}{c^2}$, where $\gamma = \dfrac{1}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}}$

The coordinates transform according to the Lorentz transformation. S' is the moving frame ($\vec{v}_0 = v_0\hat{z}$). S is the stationary frame.

Lorentz transformation:

$$\left.\begin{aligned} x' &= x \\ y' &= y \\ z' &= \gamma(z-v_0t) \end{aligned}\right\}\ \text{obtain by inspection of equations}$$

also $t' = \gamma\left(t - \dfrac{vz}{c^2}\right)$ less obvious

$V, \vec{A}, \vec{E}$, and $\vec{B}$ obey related transformations
↳ to be determined.

Motivating Question: Can you get magnetism / Maxwell's equations from the Lorentz transformation and Coulomb's law?

if yes .... other question: Is there a $\vec{B}$ equivalent in Gravity?

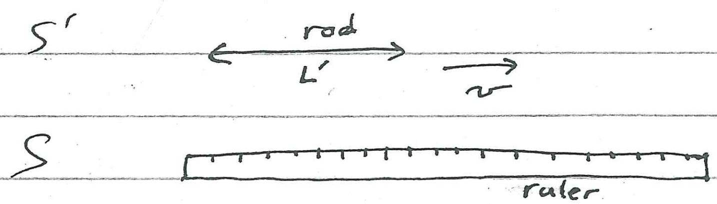
Einstein's Postulates for Special Relativity:

1 – The principle of relativity: The laws of physics are the same in all inertial reference frames.
inertial frame = coordinate system at constant velocity in a rest frame. ← frame in which you cannot tell that you are accelerating

2 – Universal speed of light: The speed of light in vacuum is the same in all inertial reference frames, regardless of the motion of the source.

Brief Review:

a – Length contraction: consider a rod of length $L' = L_0$ in the S' frame moving at $\vec{v} = v\hat{x}$



How long is the rod in S?
(simultaneous measurement of positions of both ends in S)

$$\Delta L' = \Delta z' = z_2' - z_1'$$
$$= \gamma(z_2 - vt) - \gamma(z_1 - vt)$$
$$= \gamma(z_2 - z_1)$$
$$= \gamma \, \Delta z$$

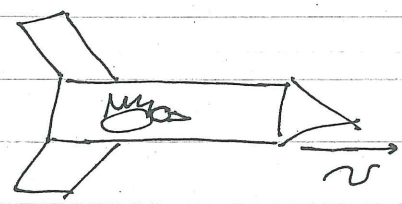
$\hookrightarrow \quad \Delta z = \frac{1}{\gamma} \Delta z' \implies L = \frac{1}{\gamma} L' \Rightarrow L = \frac{1}{\gamma} L_0$

The rod is a factor $1/\gamma$ shorter in S.

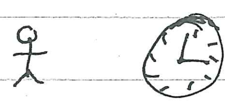
b. <u>Time dilation</u>

Astronauts on spaceship S' travelling with velocity $\vec{v} = v\hat{z}$ are preparing a turkey for Thanksgiving. On the spaceship S', they cook the turkey for $\Delta T' = T_0$. How long does the turkey cook for in your frame S? (Earth's)

S'



S



$$\Delta T' = t_2' - t_1' = \gamma\left(t_2 - \frac{v}{c^2}\cancelto{vt_2}{z_2}\right) - \gamma\left(t_1 - \frac{v}{c^2}\cancelto{vt_1}{z_1}\right)$$
$$= \gamma t_2 \left(1 - \frac{v^2}{c^2}\right) - \gamma t_1 \left(1 - \frac{v^2}{c^2}\right)$$
$$= \gamma (t_2 - t_1) \underbrace{\left(1 - \frac{v^2}{c^2}\right)}_{1/\gamma^2} = \frac{\Delta T}{\gamma}$$

$\implies \Delta T = \gamma \, \Delta T' \implies T = \gamma T_0$

The turkey takes a factor of γ longer to cook in S.

Lorentz Transformation in Matrix Form

$ct' = \gamma(ct - \frac{v}{c}z)$
$x' = x$
$y' = y$
$z' = \gamma(z - \frac{v}{c}ct)$

$\beta = \frac{v}{c}$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}}_{\Lambda(\vec{v}=v\hat{z})^{\mu}{}_{\nu}} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$\mu \leftarrow$ line
$\nu \leftarrow$ column

S' travelling with velocity $\vec{v} = v\hat{z}$ in S

4-vectors

Contravariant vectors: a^μ

example: position 4-vector $x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$

Covariant vectors: a_μ

example: position 4-vector $x_\mu = (x_0, x_1, x_2, x_3)$
$= (-ct, x, y, z)$

in general $a_\mu = (-a^0, a^1, a^2, a^3)$

How do you transform a contravariant 4-vector from one reference frame to another?

$$\underset{\text{any contravariant 4-vector}}{a^{\mu'}} = \sum_{\nu=0}^{3} \Lambda(\vec{v})^{\mu}{}_{\nu}\, a^{\nu} \equiv \underbrace{\Lambda(\vec{v})^{\mu}{}_{\nu}\, a^{\nu}}_{\text{Einstein notation: implied summation over repeated index}}$$

line

column

$$a^{\mu\prime} = \Lambda^{\mu}_{\ \nu}\, a^{\nu} = a^{\nu}\, \Lambda^{\mu}_{\ \nu}$$

$$\Leftrightarrow \begin{pmatrix} a^{0\prime} \\ a^{1\prime} \\ a^{2\prime} \\ a^{3\prime} \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}}_{\Lambda(\vec{v} = v\hat{z})^{\mu}_{\ \nu}} \begin{pmatrix} a^{0} \\ a^{1} \\ a^{2} \\ a^{3} \end{pmatrix}$$

note: $a^{\mu} = \Lambda(-\vec{v})^{\mu}_{\ \nu}\, a^{\nu\prime}$

$$\Lambda(\vec{v})^{\mu}_{\ \nu}\, \Lambda(-\vec{v})^{\nu}_{\ \eta} = \delta^{\mu}_{\ \eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

unsurprisingly $\left[\Lambda(\vec{v})^{\mu}_{\ \nu}\right]^{-1} = \Lambda(-\vec{v})^{\mu}_{\ \nu}$

Scalar product:

$$(a)^2 = a_{\mu} a^{\mu} = a_0 a^0 + a_1 a^1 + a_2 a^2 + a_3 a^3$$
$$= -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2$$

$$ab = a_{\mu} b^{\mu}$$
$$= -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

in the same frame

the scalar product of two 4-vectors is Lorentz invariant

(i.e. scalar product does not depend on your reference frame)

Q: How do covariant 4-vectors transform?

$$a'_\mu = \Lambda(-\vec{v})_\mu^{\ \nu} \, a_\nu$$

line

column

$$\underbrace{(a_0', a_1', a_2', a_3')}_{(-a^{0\prime},\ a^{1\prime},\ a^{2\prime},\ a^{3\prime})} = \underbrace{(a_0, a_1, a_2, a_3)}_{(-a^0,\ a^1,\ a_2,\ a_3)} \begin{pmatrix} \gamma & 0 & 0 & +\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

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Proof of the Lorentz invariance of the scalar product:

$$a'b' = a'_\mu b^{\mu\prime} = \underbrace{\Lambda(-\vec{v})_\mu^{\ \nu} a_\nu \, \Lambda(\vec{v})^\mu_{\ \eta}}_{\delta^\nu_\eta} b^\eta$$

$$= \delta^\nu_\eta \, a_\nu b^\eta = a_\eta b^\eta = ab$$
$$= a_\nu b^\nu$$

The metric tensors $g^{\mu\nu}$ and $g_{\mu\nu}$ allow one to switch between contravariant and covariant 4-vectors forms.

$$a^\mu = g^{\mu\nu} a_\nu \quad \text{and} \quad b_\mu = g_{\mu\nu} b^\nu$$

For special relativity: $g^{00} = -1$, $g^{ii} = 1$ $\quad i = 1,2,3$

$g^{0i} = 0$

$g^{ij} = 0 \quad i \neq j$

also $g^{\mu\nu} = g^{\nu\mu}$

$$g^{\mu\nu} g_{\nu\eta} = g^\mu_{\ \eta} = \delta^\mu_{\ \eta}$$

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$