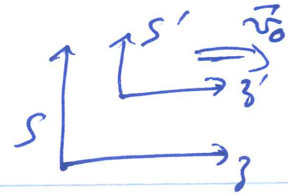

Tuesday, December 5, 2017

Q: How do contravariant 4-vectors transform?

line

$$\underline{A}: \quad a^{\mu'} = \Lambda(\vec{v})^{\mu}{}_{\nu}\, a^{\nu} = \underbrace{\begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}}_{\Lambda(\vec{v}_0 = v_0 \hat{z})} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = \begin{pmatrix} a^{0'} \\ a^{1'} \\ a^{2'} \\ a^{3'} \end{pmatrix}$$

column

Q: How do covariant 4-vectors transform?

line

$$A: \quad a_{\mu}' = \Lambda(-\vec{v}_0)_{\mu}{}^{\nu}\, a_{\nu} = \underbrace{(a_0, a_1, a_2, a_3)}_{(-a^0, a^1, a^2, a^3)} \begin{pmatrix} \gamma & 0 & 0 & +\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

column

$$(a_0', a_1', a_2', a_3') = (-a^{0'}, a^{1'}, a^{2'}, a^{3'})$$

Proof of the Lorentz invariance of the scalar product:

$$a'b' = a_{\mu}' b^{\mu'} = \underbrace{\Lambda(-\vec{v}_0)_{\mu}{}^{\nu}\, a_{\nu}\, \Lambda(\vec{v}_0)^{\mu}{}_{\eta}}_{\delta^{\nu}{}_{\eta}}\, b^{\eta} = \delta^{\nu}{}_{\eta}\, a_{\nu} b^{\eta}$$

$$= a^{\mu'} b_{\mu}'$$

$$= a_{\eta} b^{\eta}$$

$$= ab$$

Metric tensor

The metric tensors $g^{\mu\nu}$ and $g_{\mu\nu}$ allow one to switch between contravariant and covariant 4-vector forms.

$$a^{\mu} = g^{\mu\nu} a_{\nu} \quad \text{and} \quad b_{\mu} = g_{\mu\nu} b^{\nu}$$

for special relativity: $[g^{\mu\nu}] = [g_{\mu\nu}] = \underbrace{\text{“}\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}\text{”}}_{\text{“East coast metric”}}$

also $g^{\mu\nu} = g^{\nu\mu}$

$$g^{\mu\nu} g_{\nu\eta} = g^{\mu}{}_{\eta} = \delta^{\mu}{}_{\eta}$$

Examples of 4-vectors

4-position vector : $x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ \vec{r} \end{pmatrix}$

4-velocity : $v^\mu = \gamma \frac{d}{dt} x^\mu = \begin{pmatrix} \gamma c \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix} = \begin{pmatrix} \gamma c \\ \gamma \vec{v} \end{pmatrix}$

(object with velocity $\vec{v}$ in S)

$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$\boxed{\Delta \; (\vec{v}_0 = v_0 \hat{z}) \text{ with } \vec{v}_0 \neq \vec{v}}$!

$= \frac{d}{d\tau} x^\mu$ where $\tau = \sqrt{1-\frac{v^2}{c^2}}\, t = \frac{t}{\gamma}$

= proper time

note: $\boxed{v_\mu v^\mu = -c^2}$

4-acceleration : $\Gamma^\mu = \frac{d}{d\tau} v^\mu = \begin{pmatrix} \gamma \dot{\gamma} c \\ \gamma \dot{\gamma} \vec{v} + \gamma^2 \vec{a} \end{pmatrix}$

$\frac{d}{d\tau} v_\mu v^\mu = 0$

$\Leftrightarrow 2 v_\mu \frac{d}{d\tau} v^\mu = 0$

note: $\Gamma^\mu v_\mu = 0$

rest mass

4-momentum : $p^\mu = m_0 v^\mu = \begin{pmatrix} m_0 \gamma c \\ m_0 \gamma \vec{v} \end{pmatrix}$

3 velocity

$m_{\text{relativity}} = \gamma m_0$

$= \begin{pmatrix} E/c \\ \gamma \vec{p} \end{pmatrix}$

relativistic kinetic energy + rest energy

3-momentum

$$p_\mu p^\mu = -\frac{E_{rel}^2}{c^2} + \gamma^2 \vec{p}^{\,2}$$

$\vec{p}_{rel} = \gamma \vec{p}$
$= m_0 \gamma \vec{v}$
$= m_{rel} \vec{v}$

$$= -\frac{E_{rel}^2}{c^2} + \vec{p}_{rel}^{\,2}$$

$$= -m_0^2 \gamma^2 c^2 + m_0^2 \gamma^2 \vec{v}^{\,2}$$

$$= -m_0^2 \cancel{\gamma^2} c^2 \cancel{\left(1 - \frac{\vec{v}^{\,2}}{c^2}\right)}$$

$$\Rightarrow \boxed{p_\mu p^\mu = -m_0 c^2}$$

Thus $-\frac{E_{rel}}{c^2} + \vec{p}_{rel}^{\,2} = -m_0 c^2$

$$\Leftrightarrow \boxed{E_{rel}^2 - \vec{p}_{rel}^{\,2} c^2 = m_0^2 c^4}$$

$$\Rightarrow \boxed{E_{rest} = m_0 c^2 \quad E_0}$$

if $\vec{p}_{rel} = 0$ (no motion at rest)

for photon: $m_0 = 0$

$E_{rel}^2 = \vec{p}_{rel}^{\,2} c^2$

$\Rightarrow \boxed{E_{rel} = |\vec{p}_{rel}| c}$

see 09/28/2017 lecture

4-divergence:

$$\vec{\nabla} = (\partial_x, \partial_y, \partial_z) \rightarrow \frac{\partial}{\partial x^\mu} = \partial_\mu = \left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

(covariant)

note: missing "−" sign

Equivalent "Laplacian", i.e. D'Alembertian: $\Box^2 = \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x^\mu} = \partial^\mu \partial_\mu$

$$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$p_\mu p^\mu = -\frac{E_{rel}^2}{c^2} + \gamma^2 \vec{p}^{\,2}$$

$\vec{p}_{rel} = \gamma \vec{p}$
$= m_0 \gamma \vec{v}$
$= m_{rel} \vec{v}$

$$= -\frac{E_{rel}^2}{c^2} + \vec{p}_{rel}^{\,2}$$

$$= -m_0^2 \gamma^2 c^2 + m_0^2 \gamma^2 \vec{v}^{\,2}$$

$$= -m_0^2 \cancel{\gamma^2} c^2 \cancel{\left(1 - \frac{\vec{v}^{\,2}}{c^2}\right)}$$

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$$\Leftrightarrow \boxed{E_{rel}^2 - \vec{p}_{rel}^{\,2} c^2 = m_0^2 c^4}$$

$$\Rightarrow \boxed{\begin{array}{l} E_{rest} = m_0 c^2 \\ E_0 \end{array}}$$

if $\vec{p}_{rel} = 0$ (no motion at rest)

for photon: $m_0 = 0$

$E_{rel}^2 = \vec{p}_{rel}^{\,2} c^2$

$\Leftrightarrow \boxed{E_{rel} = |\vec{p}_{rel}|\, c}$

see 09/28/2017 lecture

4-divergence:

$$\vec{\nabla} = (\partial_x, \partial_y, \partial_z) \to \frac{\partial}{\partial x^\mu} =$$

(covariant)

Why the counter-intuitive assignment?

$$\frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial \phi}{\partial x^\nu} \underbrace{\frac{\partial x^\nu}{\partial x^{\mu'}}}_{\Lambda(-v_0)^\nu{}_\mu}$$

since $x^{\mu'} = \Lambda(\vec{v}_0)^\mu{}_\nu\, x^\nu$

$\hookrightarrow x^\nu = \Lambda(-\vec{v}_0)^\nu{}_\mu\, x^{\mu'}$

Equivalent "Laplacian", i.e. D'Alembertian:

4-gradient : $\partial_\mu \phi = \left(\frac{1}{c} \frac{\partial}{\partial t} \phi , \vec{\nabla} \phi \right)$

4-current density :

in 3D : $\vec{J} = \rho \vec{v} \longrightarrow J^\mu \overset{?}{=} \rho v^\mu$

yes, but only if we use the rest charge density (often called the "proper charge density").

$$J^\mu = \rho_0 v^\mu \quad \text{with } \rho_0 = \frac{Q}{V_0} , \quad \rho = \frac{Q}{V}$$

with $V = \frac{V_0}{\gamma}$ (length contraction)

$$\hookrightarrow \rho = \gamma \rho_0$$

Thus $J^\mu = \rho_0 v^\mu = \begin{pmatrix} \gamma \rho_0 c \\ \gamma \rho_0 \vec{v} \end{pmatrix} = \begin{pmatrix} \rho c \\ \vec{J} \end{pmatrix}$

Very important: Q is the same in all reference frames. charge is a Lorentz invariant and is always conserved.

Relativistic Continuity Equation :

in 3D : $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \Leftrightarrow \frac{\partial c\rho}{c \partial t} + \frac{\partial}{\partial x} J_x + \frac{\partial}{\partial y} J_y + \frac{\partial}{\partial z} J_z = 0$

$$\Leftrightarrow \boxed{\partial_\mu J^\mu = 0}$$

Lorentz Invariant Formulation of Electrodynamics

4-vector Potential (or 4-potential)

Recall that in the Lorentz gauge, Maxwell's equations take the form:

$$\begin{cases} \Box^2 V = -\rho/\varepsilon_0 \\ \Box^2 \vec{A} = -\mu_0 \vec{J} \end{cases} \Longleftrightarrow \begin{cases} \Box^2 \dfrac{V}{c} = -\dfrac{\rho}{\varepsilon_0 c} = -\dfrac{\mu_0}{\varepsilon_0 \mu_0 c}\rho = -\mu_0 c\rho \\ \Box^2 \vec{A} = -\mu_0 \vec{J} \end{cases}$$

J^μ

This suggests that the 4-potential is $A^\mu = \begin{pmatrix} V/c \\ \vec{A} \end{pmatrix} = \begin{pmatrix} V/c \\ A_x \\ A_y \\ A_z \end{pmatrix}$

Thus we have

$$\boxed{\Box^2 A^\mu = -\mu_0 J^\mu \Leftrightarrow \partial_\mu \partial^\mu A^\nu = -\mu_0 J^\nu}$$

Q: Can we construct 4-vectors for $\vec{E}$ & $\vec{B}$?

A: No! since $\vec{E}$-field in one frame (static charge) will acquire a $\vec{B}$-field in a moving frame $\rightarrow$ 4-vector does not have enough components.
$\Rightarrow$ We need a mathematical object that contains both $\vec{E}$ & $\vec{B}$.

Electromagnetic Field Tensor $F^{\mu\nu}$

(generally referred to as "F-mu-nu")

The EM field is described in ~~Lorentz~~ Minkowski space 4-space with an anti-symmetric 4-tensor:

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

μ: line, ν: column

F^{12}

right hand rule mnemonic

note: $F^{\mu\nu} = -F^{\nu\mu}$

stopped here

<u>How does $F^{\mu\nu}$ transform?</u>

$$F^{\mu\nu\prime} = \Lambda(\vec{v}_0)^{\mu}_{\ \gamma}\, \Lambda(\vec{v}_0)^{\nu}_{\ \sigma}\, F^{\gamma\sigma}$$

also

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

<u>Note:</u> $F^{\mu\nu} F_{\mu\nu} = 2\left(\vec{B}^2 - \dfrac{\vec{E}^2}{c^2}\right)$ [true in all reference frames]

Also, it turns out that

$$\boxed{F^{\mu\nu} = \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}}$$

equivalent to $\begin{cases} \vec{E} = -\vec{\nabla} V - \dfrac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$