Thursday, December 7, 2017

Electromagnetic Field Tensor $F^{\mu\nu}$

The EM field is described with anti-symmetric 4-tensor:

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

line

column

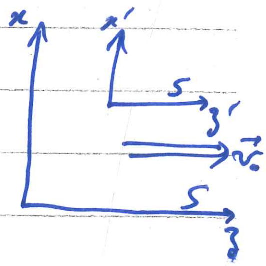
F^{12}

F^{21}

note: $F^{\mu\nu} = -F^{\nu\mu}$

right hand rule mnemonic

Q: How does $F^{\mu\nu}$ transform? (hint: it's doubly contravariant.)



$$F^{\mu\nu\prime} = \Lambda(\vec{v}_0)^{\mu}_{\ \eta}\, \Lambda(\vec{v}_0)^{\nu}_{\ \sigma}\, F^{\eta\sigma}$$

also $$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

Note: $$F^{\mu\nu} F_{\mu\nu} = 2\left(\vec{B}^2 - \frac{\vec{E}^2}{c^2}\right)$$ [true in all reference frames]

Also, it turns out that

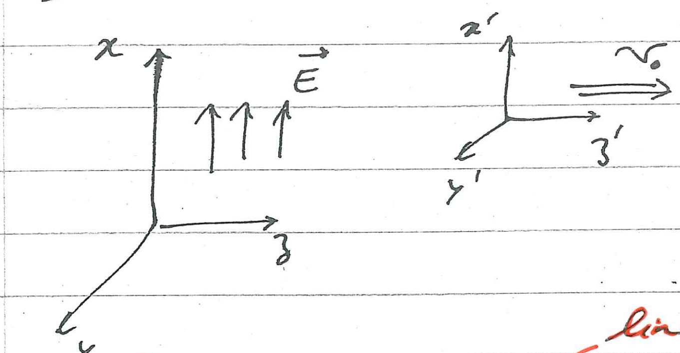
$$\boxed{F^{\mu\nu} = \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}}$$

equivalent to $\begin{cases} \vec{E} = -\nabla V - \dfrac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$

Example: – In frame S, there exists a constant E-field: $\vec{E} = E_0\hat{x}$ and no $\vec{B}$-field

– Frame S' moves with velocity $\vec{v}_0 = v_0\hat{z}$ in the S frame.

Q: What are $\vec{E}$ & $\vec{B}$ in the S' frame?



$$\beta = \frac{v_0}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\frac{v_0^2}{c^2}}}$$

recall: $\Lambda(\vec{v}_0)^{\sigma}{}_{\nu}$ — line (σ), column (ν) — symmetric

line, column

$$F'^{\mu\nu} = \Lambda(\vec{v}_0)^{\mu}{}_{\xi}\,\Lambda(\vec{v}_0)^{\nu}{}_{\sigma}\,F^{\xi\sigma}$$

$$\begin{bmatrix} 0 & E_x'/c & E_y'/c & E_z'/c \\ -E_x'/c & 0 & B_z' & -B_y' \\ -E_y'/c & -B_z' & 0 & B_x' \\ -E_z'/c & B_y' & -B_x' & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}\begin{bmatrix} 0 & E_x/c & \cancel{E_y/c}\ 0 & \cancel{E_z/c}\ 0 \\ -E_x/c & 0 & \cancel{B_z}\ 0 & \cancel{-B_y}\ 0 \\ \cancel{-E_y/c}\ 0 & \cancel{-B_z}\ 0 & 0 & \cancel{B_x}\ 0 \\ \cancel{-E_z/c}\ 0 & \cancel{B_y}\ 0 & \cancel{-B_x}\ 0 & 0 \end{bmatrix}}_{\text{do first}}\begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

do second

$$\begin{bmatrix} 0 & E_x'/c & E_y'/c & E_z'/c \\ -E_x'/c & 0 & B_z' & -B_y' \\ -E_y'/c & -B_z' & 0 & B_x' \\ -E_z'/c & B_y' & -B_x' & 0 \end{bmatrix} = \begin{bmatrix} 0 & \gamma E_x/c & 0 & 0 \\ -E_x/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\beta\gamma E_x}{c} & 0 & 0 \end{bmatrix}\begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

$$\begin{bmatrix} 0 & E_x'/c & E_y'/c & E_z'/c \\ -E_x'/c & 0 & B_z' & -B_y' \\ -E_y'/c & -B_z' & 0 & B_x' \\ -E_z'/c & B_y' & -B_x' & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \gamma E_x/c & 0 & 0 \\ -\gamma E_x/c & 0 & 0 & \frac{\beta\gamma E_x}{c} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\beta\gamma E_x}{c} & 0 & 0 \end{bmatrix}}_{\text{note: still anti-symmetric}}$$

$$\Rightarrow \begin{cases} E_x' = \gamma E_x \\ B_y' = -\frac{\beta\gamma E_x}{c} \end{cases}$$ → agrees with right hand rule

More generally, if $\vec{B} = 0$ in S, then $\boxed{\vec{B}' = -\frac{1}{c^2}\vec{v} \times \vec{E}'}$

if $\vec{E} = 0$ in S, then $\boxed{\vec{E}' = \vec{v} \times \vec{B}'}$

Maxwell's Equations in 4-space / Minkowski space notation

definition: dual tensor of $F^{\mu\nu}$: $G^{\mu\nu}$ $\left(\frac{\vec{E}}{c} \to \vec{B}, \vec{B} \to -\vec{E}/c\right)$

line

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & +E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

Column

$$\begin{bmatrix} 0 & E_x'/c & E_y'/c & E_z'/c \\ -E_x'/c & 0 & B_z' & -B_y' \\ -E_y'/c & -B_z' & 0 & B_x' \\ -E_z'/c & B_y' & -B_x' & 0 \end{bmatrix} =$$

$$F^{\mu\nu'} = \Lambda^\mu{}_0 \Lambda^\nu{}_1 \underline{F^{01}}$$

$$+ \Lambda^\mu{}_1 \Lambda^\nu{}_0 \underline{F^{10}} \rightarrow \text{only non-zero terms}$$

$$= \left(\Lambda^\mu{}_0 \Lambda^\nu{}_1 - \Lambda^\mu{}_1 \Lambda^\nu{}_0\right) \frac{E_x}{c}$$

$$\left.\begin{aligned} F^{\mu 1'}:\ & F^{01'} = \gamma E_x/c \\ & F^{31'} = -\beta\gamma E_x/c \\ F^{1\nu'}:\ & F^{10'} = -\gamma E_x/c \\ & F^{13'} = \beta\gamma E_x/c \end{aligned}\right\} \text{only 4 non-zero terms}$$

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Column

4 vector equations
$\Rightarrow$ 8 scalar equations

$$\partial_\nu F^{\mu\nu} = \partial_\nu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \partial^\mu \underbrace{\partial_\nu A^\nu}_{\substack{=0 \\ \text{Lorentz Gauge}}} - \underbrace{\partial_\nu \partial^\nu}_{\Box^2} A^\mu = -\Box^2 A^\mu = \mu_0 J$$

Maxwell's Equations:

$$\boxed{\frac{\partial}{\partial x^\nu} F^{\mu\nu} = \partial_\nu F^{\mu\nu} = \mu_0 J^\mu \quad \text{and} \quad \partial_\nu G^{\mu\nu} = 0}$$

4 equations (Gauss + Ampère's)

4 equations (no mag. monopoles, Faraday)

Example: for $\mu = 0$

$$\frac{\partial F^{0\nu}}{\partial x^\nu} = \mu_0 J^0 \iff \frac{\partial}{\partial(ct)} \cancelto{0}{F^{00}} + \frac{\partial}{\partial x} \cancelto{E_x/c}{F^{01}} + \frac{\partial}{\partial y} \cancelto{E_y/c}{F^{02}} + \frac{\partial}{\partial z} \cancelto{E_z/c}{F^{03}} = \mu_0 c \rho$$

$$\iff \partial_x E_x + \partial_y E_y + \partial_z E_z = \underbrace{\mu_0 c^2}_{1/\varepsilon_0} \rho$$

$$\iff \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(Gauss's Law)}$$

Q: Why doesn't gravity have a "magnetic field" associated with it?

A: (short answer) Charge is a Lorentz invariant. mass is not a Lorentz invariant.

Lorentz Force Law

The Lorentz force law is correct according to relativity

$$\vec{F} = \underbrace{q\vec{E} + q\vec{v}\times\vec{B}}_{\text{incorporates Special Relativity}}$$

$\cancel{m\vec{a}}$ does not incorporate Special Relativity

$$\vec{F} = \frac{d\vec{p}_{rel}}{dt} \quad \text{where } \vec{p}_{rel} = \frac{m_0\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

The equations of motion of a charge in $\vec{E}$ and $\vec{B}$ fields are given by

$$\frac{d\vec{p}_{rel}}{dt} = q\vec{E} + q\vec{v}\times\vec{B}$$

Minkowski force : $K^\mu = \frac{d}{d\tau}p^\mu = m_{rest}\Gamma^\mu$

recall: $p^\mu = \begin{pmatrix} \gamma\, m_{rest}\, c \\ \gamma\, m_{rest}\,\vec{v} \end{pmatrix} = m_{rest}\, v^\mu$

Lorentz Force law becomes

$$\boxed{K^\mu = q F^{\mu\nu} v_\nu}$$

Q: Can you formulate a Lagrangian description for relativistic particle dynamics i.e. Lorentz Force Law? A: Yes

Q: Can you formulate a Lagrangian description for electrodynamics, i.e. Maxwell's equations? A: Yes: PHYS 610