Tuesday, September 12, 2017

local Conservation of charge → continuity equation: $\boxed{\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}}$

$\hookrightarrow \frac{\partial \rho}{\partial t} \neq 0 \iff \vec{\nabla}\cdot\vec{J} \neq 0$

The problem with Ampère's Law

$$\vec{\nabla}\cdot\left\{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}\right\} \Rightarrow \underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{\text{div.curl}=0} = \mu_0 \underbrace{\vec{\nabla}\cdot\vec{J}}_{\neq 0}$$

generally in electrodynamics

⇒ Ampère's law cannot be correct for time-varying $\vec{E}$ & $\vec{B}$ fields and ρ & $\vec{J}$ distributions.

note: $\mu_0\vec{\nabla}\cdot\vec{J} = -\mu_0\frac{\partial \rho}{\partial t} = -\mu_0\varepsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E})$

insert Gauss's law

$$= -\mu_0\varepsilon_0\vec{\nabla}\cdot\left(\frac{\partial \vec{E}}{\partial t}\right)$$

If we add an extra term to Ampère's law, then the problem is resolved:

Maxwell's modification to Ampère's law

$$\vec{\nabla}\cdot\left\{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}\right\}$$

$$\Rightarrow \underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{=0} = \underbrace{\mu_0\vec{\nabla}\cdot\vec{J}}_{\cancel{-\mu_0\varepsilon_0\vec{\nabla}\cdot\left(\frac{\partial\vec{E}}{\partial t}\right)}} + \cancel{\varepsilon_0\mu_0\vec{\nabla}\cdot\left(\frac{\partial\vec{E}}{\partial t}\right)}$$

$$\Rightarrow \quad 0 = 0$$

Ampère's improved Law : $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}}$

Maxwell's Equations in Vacuum

1) $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ — Gauss's Law

2) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ — Faraday's Law

3) $\vec{\nabla}\cdot\vec{B} = 0$ — no magnetic monopoles law

4) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ — Ampère's improved law

Maxwell's equations quiz on Tuesday, September 19, 2017

5) $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

conservation of charge*

this eq. is already inside 1) – 4)

$\nabla\cdot(4)$ with (1) gives (5).

Symmetry : time-dependent $\vec{E} \longrightarrow \vec{B}$

time-dependent $\vec{B} \longrightarrow \vec{E}$

Asymmetry : $\vec{\nabla}\cdot\vec{B} = ??\ \neq 0 \rightarrow$ a magnetic charge would fit into the theory but none have been found (reproducibly).

All of electrodynamics, optics, special relativity, microwave electronics, antenna theory, diffraction, synchrotron radiation, etc... follows from Maxwell's equations... + boundary conditions → need E&M in matter.

Displacement Current

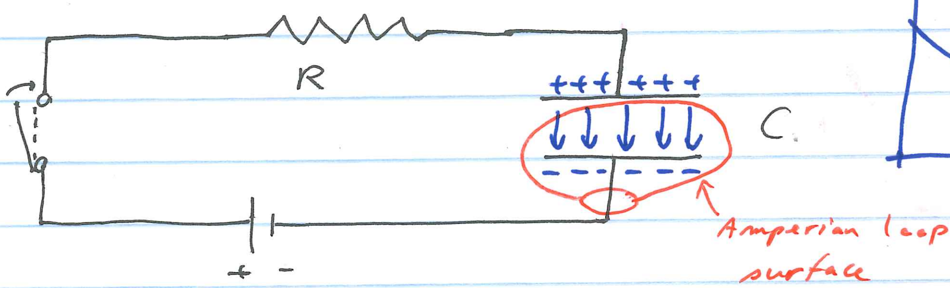
The term "$\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$" is referred to as the displacement current:

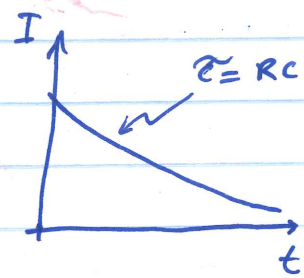
$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$\Rightarrow$ Ampère's law (improved): $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \vec{J}_D$

Thinking of a time-varying $\vec{E}$-field as a "current" can be physically useful:

Consider a charging capacitor:





- physics is clear

- Math does not work with original Ampère's law.

Review of Electrostatics & Magnetostatics in Matter

Electrostatics in Matter

In a polarizable material (dielectric) with polarization per unit volume $\vec{P}(\vec{r})$ (or dipole moment per unit volume) there are bound charges:

$\hat{n}$ points out of surface ($\perp$)

bound surface charges : $\sigma_b(\vec{r}) = \vec{P}(\vec{r})\cdot\hat{n}$

bound volume charges : $\rho_b(\vec{r}) = -\vec{\nabla}\cdot\vec{P}(\vec{r})$

There are also <u>free</u> charges (controlled by experimenter): $\rho_f(\vec{r})$

total charge density: $\rho_{total} = \rho_f + \rho_b + \underbrace{\sigma_b}_{\text{only on surface/boundary of material}}$

We define the "electric displacement" as $\boxed{\vec{D} = \varepsilon_0\vec{E} + \vec{P}}$

$\vec{D}$ obeys Gauss's law :

$$\left\{\begin{array}{l} \boxed{\vec{\nabla}\cdot\vec{D} = \rho_f} \\ \boxed{\displaystyle\oint_{\substack{\text{closed}\\ \text{surface}}} \vec{D}\cdot d\vec{s} = q_{f,\,\text{enclosed}}} \end{array}\right.$$

⚠ note: $\boxed{\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P} \neq 0}$ in electrostatics

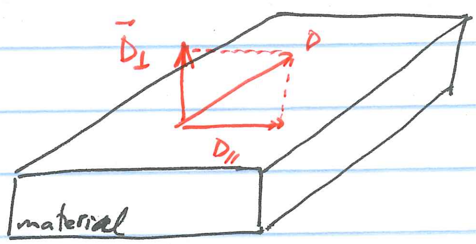
In a linear dielectric: $\boxed{\vec{P} = \varepsilon_0 \chi_e \vec{E}}$ where χ_e = electric susceptibility

permittivity of the material: $\varepsilon = \varepsilon_0 (1+\chi_e)$

$\vec{D} = \varepsilon \vec{E}$

$\hookrightarrow \boxed{\vec{\nabla} \cdot \vec{E} = \dfrac{\rho_f}{\varepsilon}}$

boundary conditions:



$D_{\perp, out} - D_{\perp, in} = \sigma_f$

$\vec{D}_{\parallel, out} - \vec{D}_{\parallel, in} = \vec{P}_{\parallel, out} - \vec{P}_{\parallel, in}$

$\Longleftrightarrow \boxed{\vec{E}_{\parallel, out} = \vec{E}_{\parallel, in}}$ used the most in E & M.

Magnetostatics in matter

In magnetized materials, $M(\vec{r})$ = magnetization per unit volume
= magnetic dipole moment per unit volume

There are bound currents:

bound surface current density: $\vec{K}_b(\vec{r}) = \vec{M}(\vec{r}) \times \hat{n}$

bound volume current density: $\vec{J}_b(\vec{r}) = \vec{\nabla} \times \vec{M}(\vec{r})$

there can also be free current (density): $\vec{J}_f(\vec{r})$ controlled by experimenter

Total current density: $\vec{J}_{total} = \vec{J}_f + \vec{J}_b + \underbrace{\vec{K}_b}_{\text{only on surface of the material}}$

We define the <u>Auxilary field</u> $\vec{H}$ as $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

Ampère's law (magnetostatics) becomes

$$\boxed{\vec{\nabla}\times\vec{H} = \vec{J}_f \quad \text{or} \quad \oint_{loop} \vec{H}\cdot d\vec{l} = I_{f,\,enclosed}}$$

⚠ <u>note</u>: $\boxed{\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M} \neq 0}$

In a <u>linear medium</u> : $M = \chi_m H$ where χ_m = magnetic susceptibility

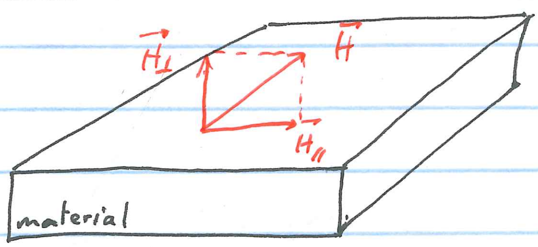
magnetic permeability of a material $= \mu = \mu_0(1+\chi_m)$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu\vec{J}_f}$$

<u>note</u>: $\vec{H} = \frac{1}{\mu_0}\vec{B} - \chi_m\vec{H}$

$$\Leftrightarrow \vec{H}(1+\chi_m) = \frac{1}{\mu_0}\vec{B} \Leftrightarrow \boxed{\vec{H} = \vec{B}/\mu}$$ for a linear medium

<u>boundary conditions</u> :



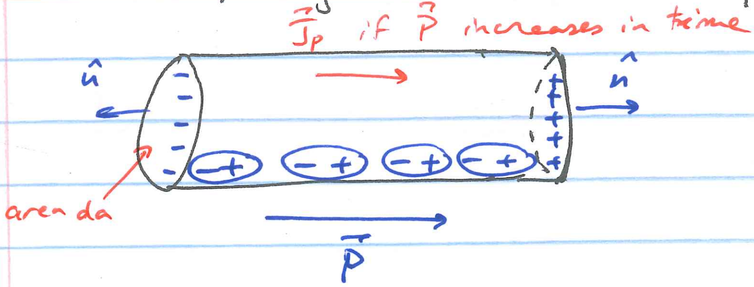
$$H_{\perp,out} - H_{\perp,in} = -(M_{\perp,out} - M_{\perp,in})$$

$$\boxed{B_{\perp,out} = B_{\perp,in}}$$ used the most in E&M

also $\vec{H}_{\parallel,out} - \vec{H}_{\parallel,in} = \vec{K}_f \times \hat{n}$

Continuity equation in matter

If we consider a time-dependent $\overbrace{\text{polarization}}^{\vec{P}(\vec{r},t)}$, then a change in polarization produces a change in bound charge, which according to the continuity equation produces a bound "polarization current": $\vec{J}_p$



current for increasing $\vec{P}$:

$$dI = \underbrace{\frac{\partial \sigma_b}{\partial t}\, da = \frac{\partial P}{\partial t}\, da}_{\sigma_b = \vec{P}\cdot\hat{n}}$$

$$\Rightarrow \quad \boxed{\vec{J}_p = \frac{\partial \vec{P}}{\partial t}} \qquad \frac{dI}{da}$$

Stopped here

Thus: $\vec{\nabla}\cdot\vec{J}_p = \vec{\nabla}\cdot\frac{\partial \vec{P}}{\partial t} = \frac{\partial}{\partial t}\underbrace{\left(\vec{\nabla}\cdot\vec{P}\right)}_{-\rho_b(\vec{r})} = -\frac{\partial}{\partial t}\rho_b$

↳ continuity equation for bound charge: $\boxed{\frac{\partial}{\partial t}\rho_b + \vec{\nabla}\cdot\vec{J}_p = 0}$

thus, the <u>total current density</u> is thus

$$\vec{J}_{total} = \vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p \quad + \underbrace{\vec{K}_b}_{\text{(bound surface current)}}$$

<u>Improved Ampère's law</u>: $\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}_f + \underbrace{\vec{J}_b}_{\vec{\nabla}\times\vec{M}} + \vec{J}_p\right) + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$

$$\Rightarrow \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}\right) + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \underbrace{\vec{\nabla}\times\vec{B} - \mu_0 \vec{\nabla}\times\vec{M}}_{\mu_0 \vec{\nabla}\times\underbrace{\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right)}_{\vec{H}}} = \mu_0 \vec{J}_f + \mu_0 \underbrace{\left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t}\right)}_{\frac{\partial \vec{D}}{\partial t}}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{H} = \vec{J}_f + \underbrace{\frac{\partial \vec{D}}{\partial t}}}$$

Ampère's improved law in matter

displacement current is now $\vec{J}_D = \frac{\partial \vec{D}}{\partial t}$