last time we showed local conservation of EM + Mechanical Energy

$$\boxed{\frac{\partial}{\partial t}\left(\mathcal{E}_{Mech} + \mathcal{E}_{EM}\right) = -\vec{\nabla}\cdot\vec{S}}$$

#1

Thursday, September 21, 2017

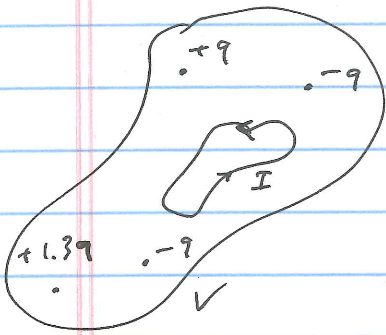
Conservation of Momentum

Just as an EM field has energy, it also carries momentum.

not surprising since photons (QM) carry momentum, but this originally a classical concept.

Maxwell Stress Tensor

Consider a collection of charges & currents in a volume V.



Q: what's the total EM force on all the charges in V?

A: $$\vec{F}_{total} = \int_V \rho(\vec{E} + \vec{v}\times\vec{B})\,d^3r = \frac{d\vec{P}}{dt}$$ momentum

$$= \int_V (\rho\vec{E} + \vec{J}\times\vec{B})\,d^3r$$

We define the force density or force per unit volume as

$$\vec{f} = \rho\vec{E} + \vec{J}\times\vec{B}$$

$\varepsilon_0 \vec{\nabla}\cdot\vec{E}$ (Gauss)

$\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0\frac{\partial\vec{E}}{\partial t}$ (Ampère's improved law)

$$\Rightarrow \vec{f} = \varepsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \Big[\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}) - \underbrace{\varepsilon_0\frac{\partial\vec{E}}{\partial t}\Big]\times\vec{B}}_{\text{target this term}}$$

we note: $\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \left(\frac{\partial \vec{E}}{\partial t}\right)\times\vec{B} + \vec{E}\times\underbrace{\left(\frac{\partial}{\partial t}\vec{B}\right)}_{-\vec{\nabla}\times\vec{E}\ \text{(Faraday)}}$

$$\Rightarrow \left(\frac{\partial \vec{E}}{\partial t}\right)\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\vec{\nabla}\times\vec{E})$$

thus $\vec{f} = \varepsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} - \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \varepsilon_0\vec{E}\times(\vec{\nabla}\times\vec{E})$

$$\vec{f} = \underbrace{-\varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})}_{-\varepsilon_0\mu_0\frac{\partial}{\partial t}\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right)} - \varepsilon_0\underbrace{\vec{E}\times(\vec{\nabla}\times\vec{E})}_{\frac{1}{2}\vec{\nabla}(E^2)-(\vec{E}\cdot\vec{\nabla})\vec{E}} + \varepsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} - \frac{1}{\mu_0}\underbrace{\vec{B}\times(\vec{\nabla}\times\vec{B})}_{\frac{1}{2}\vec{\nabla}(\vec{B}^2)-(\vec{B}\cdot\vec{\nabla})\vec{B}} + \underbrace{\frac{1}{\mu_0}(\underbrace{\vec{\nabla}\cdot\vec{B}}_{=0})\vec{B}}_{\text{this term makes equation more symmetric}}$$

$$= -\varepsilon_0\mu_0\frac{\partial}{\partial t}\vec{S}$$

$$(\vec{E}\cdot\vec{\nabla})\vec{E} = (E_x\partial_x E_x + E_y\partial_y E_x + E_z\partial_z E_x)\hat{x} + (E_x\partial_x E_y + E_y\partial_y E_y + E_z\partial_z E_y)\hat{y} + (E_x\partial_x E_z + E_y\partial_y E_z + E_z\partial_z E_z)\hat{z}$$

Thus,

$$\vec{f} = -\varepsilon_0\mu_0\frac{\partial}{\partial t}\vec{S} \underbrace{- \frac{1}{2}\vec{\nabla}\left(\varepsilon_0\vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right) + \varepsilon_0\left\{(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E}\right\} + \frac{1}{\mu_0}\left\{(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B}\right\}}_{\vec{\nabla}\cdot ?}$$

we will try to get an equation of the form

$$\frac{\partial \vec{S}}{\partial t} + \vec{\nabla}\cdot ? = 0$$

i.e. a continuity-type equation

Define : Maxwell's stress Tensor

$$\overleftrightarrow{T} = T_{ij} = \varepsilon_0 \left[E_i E_j - \frac{1}{2} \delta_{ij} \vec{E}^2 \right] + \frac{1}{\mu_0} \left[B_i B_j - \frac{1}{2} \vec{B}^2 \delta_{ij} \right]$$

tensor = matrix or 2-D vector , δ_{ij} = Kronecker delta
.... n-dimensional $= \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases}$

Divergence of a Tensor

regular 1D vector $= \left(\vec{\nabla} \cdot \overleftrightarrow{T}\right)_j = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} T_{ij} = (\partial_x, \partial_y, \partial_z) \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$

$x_1 = x,\ x_2 = y,\ x_3 = z$

One can show that (i.e. multiply out all the components)

$$\vec{\nabla} \cdot \overleftrightarrow{T} = \varepsilon_0 \left\{ (\vec{\nabla} \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{E} \right\} + \frac{1}{\mu_0} \left\{ (\vec{\nabla} \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{B} \right\} - \frac{1}{2} \vec{\nabla} \left\{ \varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0} \vec{B}^2 \right\}$$

thus $\boxed{\vec{f} = -\varepsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t} + \vec{\nabla} \cdot \overleftrightarrow{T}}$

$$\Rightarrow \vec{F}_{total}\Big|_V = \int_V \vec{f}\, d^3r = -\varepsilon_0 \mu_0 \frac{\partial}{\partial t} \int_V \vec{S}\, d^3r + \underbrace{\int_V \left(\vec{\nabla} \cdot \overleftrightarrow{T}\right) d^3r}_{}$$

"Divergence theorem for ~~te~~ tensors" $\oint_{S(V)} \underbrace{\overleftrightarrow{T} \cdot d\vec{s}}_{} = 1D \text{ vector} = \sum_j T_{ij}\, ds_j$

$$\Rightarrow \vec{F}_{total}\Big|_V = -\frac{d}{dt} \int_V \varepsilon_0 \mu_0 \vec{S}\, d^3r + \oint_{S(V)} \overleftrightarrow{T} \cdot d\vec{s}$$

What is $\overleftrightarrow{T}$?

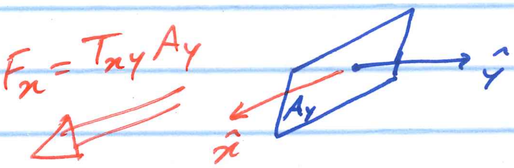
$\overleftrightarrow{T}$ has dimensions/units of energy density $\left[J/m^3 \right]$ ($J = N \cdot m$)

or (more useful) Force per unit area $\left[N/m^2 \right]$

↳ i.e. it's a "pressure".

T_{xx}, T_{yy}, and T_{zz} represent pressure in the x, y, z directions, while T_{xy} is a shear (i.e. a force per unit area in the x-direction acting on a surface with direction $\hat{y}$).

$F_x = T_{xy} A_y$



Total force on volume V:

Newton's laws

- A system of particles & fields that acts only on itself should have $\vec{F}_{total}\Big|_V = 0$

- If there are external forces acting on V, then $\vec{F}_{total}\Big|_V = \vec{F}_{external} = \frac{d}{dt} \vec{P}_{Mech}\Big|_V$

  where $\vec{P}_{Mech}\Big|_V$ is the mechanical momentum of the charge distribution.

⚠ there are no known EM forces that act directly on $\vec{E}$ & $\vec{B}$.

Thus
$$\boxed{\vec{F}_{total}\Big|_V = \frac{d}{dt} \underbrace{\vec{P}_{Mech}}_{\text{Mechanical momentum of charges/currents}}\Big|_V = -\frac{d}{dt} \underbrace{\int_V \varepsilon_0 \mu_0 \vec{S}\, d^3r}_{\text{"momentum"}} + \oint_{S(V)} \overleftrightarrow{T} \cdot d\vec{s}}$$

$\oint_{S(V)} \overleftrightarrow{T} \cdot d\vec{s}$: EM forces acting on the surface of V.

This equation suggests:

(definition) $\vec{P}_{EM} = \varepsilon_0 \mu_0 \int_V \vec{S}\, d^3r$ = E-M momentum (momentum stored in E-M fields)

$$\boxed{\vec{\wp}_{EM} = \underbrace{\varepsilon_0 \mu_0 \vec{S}}_{\varepsilon_0 \vec{E}\times\vec{B}} = \text{E-M momentum density}}$$

even static EM fields can have momentum

also, $\overleftrightarrow{T}$ = momentum per unit time through surface V.
= momentum flux or "current" per unit area.
(momentum current density)

(if we consider an infinitesimal volume)

$$\text{thus } \boxed{\frac{d}{dt}\Big(\underbrace{\vec{\wp}_{EM} + \vec{\wp}_{Mech}}_{\text{total momentum density}}\Big) = \vec{\nabla}\cdot\overleftrightarrow{T}} + \text{momentum of particles entering \& exiting local region}$$

note missing "−" sign → def of $\overleftrightarrow{T}$

" $\vec{\nabla}\cdot\overleftrightarrow{\mathbb{T}}$ "

mechanical stress tensor

Total momentum (Mechanical + EM) is a locally conserved quantity

Angular Momentum

EM fields can also have angular momentum

↳ it is also conserved (but it is origin dependent)

Angular momentum density = $\vec{\ell}_{EM} = \vec{r} \times \vec{\wp}_{EM} = \varepsilon_0 \vec{r} \times (\vec{E}\times\vec{B})$