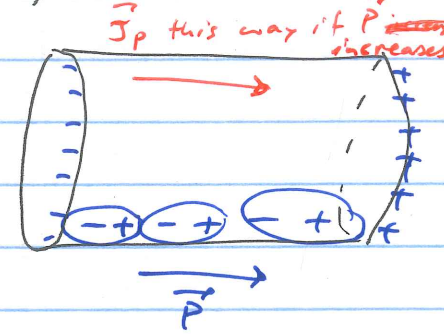
Tuesday, September 19, 2017

__Continuity Equation in Matter__ (continued) → objective: Maxwell's equations in matter.

Last time we saw that a time-dependent polarization gives rise to a polarization current $\vec{J}_p$:

$$\boxed{\vec{J}_p = \frac{\partial \vec{P}}{\partial t}}$$



$$\Rightarrow \quad \vec{\nabla}\cdot\vec{J}_p = \vec{\nabla}\cdot\left(\frac{\partial \vec{P}}{\partial t}\right) = \frac{\partial}{\partial t}\big(\underbrace{\vec{\nabla}\cdot\vec{P}}_{-\rho_b(\vec{r}) \,=\, \text{bound charge}}\big)$$

(see last Tuesday)

$$= -\frac{\partial}{\partial t}\rho_b$$

$\Rightarrow$ continuity equation for bound charge: $\boxed{\frac{\partial}{\partial t}\rho_b + \vec{\nabla}\cdot\vec{J}_p = 0}$

Thus, the __total current density__ is

$$\vec{J}_{total} = \vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p \text{ "} + \vec{K}_b\text{"}$$

(bound surface current)

__Improved Ampère's law__: (in the bulk, i.e. ignore surface)

$$\vec{\nabla}\times\vec{B} = \mu_0\big(\vec{J}_f + \underbrace{\vec{J}_b}_{\vec{\nabla}\times\vec{M}} + \vec{J}_p\big) + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}\right) + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \underbrace{\vec{\nabla}\times\vec{B} - \mu_0 \vec{\nabla}\times\vec{M}}_{\mu_0 \vec{\nabla}\times\left(\underbrace{\frac{\vec{B}}{\mu_0} - \vec{M}}_{\vec{H}}\right)} = \mu_0 \vec{J}_f + \mu_0 \underbrace{\left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t}\right)}_{\frac{\partial}{\partial t}\vec{D}}$$

$(\vec{D} = \varepsilon_0 \vec{E} + \vec{P})$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{H} = \vec{J}_f + \underbrace{\frac{\partial \vec{D}}{\partial t}}_{\text{displacement current}}}$$

Ampère's improved law in matter

displacement current is now $\vec{J}_D = \frac{\partial \vec{D}}{\partial t}$

Maxwell's Equations in Matter

1) $\vec{\nabla}\cdot\vec{D} = \rho_f$
2) $\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$
3) $\vec{\nabla}\cdot\vec{B} = 0$
4) $\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$

(2) and (3): unchanged since these do not involve bound charges & currents

$\vec{D} = \epsilon \vec{E}$, $\vec{H} = \frac{1}{\mu}\vec{B}$

Linear Medium

1) $\vec{\nabla}\cdot\vec{E} = \rho_f/\varepsilon$
2) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$
3) $\vec{\nabla}\cdot\vec{B} = 0$
4) $\vec{\nabla}\times\vec{B} = \mu \vec{J}_f + \varepsilon\mu \frac{\partial \vec{E}}{\partial t}$

Maxwell's equations in linear media

boundary conditions: (unchanged since $\vec{J}_P$ is inside material)

For linear media: medium 1 – medium 2 boundary

same as electro/magneto statics

1) $\varepsilon_1 E_{\perp,1} - \varepsilon_2 E_{\perp,2} = \sigma_f = 0$ for no free charges
2) $\vec{E}_{\parallel,1} = \vec{E}_{\parallel,2}$
3) $B_{\perp,1} = B_{\perp,2}$
4) $\frac{1}{\mu_1}\vec{B}_{\parallel,1} - \frac{1}{\mu_2}\vec{B}_{\parallel,2} = \vec{K}_f \times \hat{n} = 0$ for no free surface currents

Conservation Laws for E-M fields

We will model our approach on the continuity equation for local conservation of charge:

$$\frac{\partial \rho(\vec{r})}{\partial t} = -\vec{\nabla}\cdot\vec{J}(\vec{r})$$

A local increase or decrease in charge density is accompanied by the creation of an ingoing or outgoing current density.

Poynting's theorem: Conservation of Energy

(Griffiths p346–349)

The energy stored in an E-M field is:

$$E_{EM} = \frac{1}{2}\int_{\substack{\text{all}\\\text{space}}} \left(\varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right) d^3r$$

Q: Is E-M energy conserved? .... yes, but...

Stronger statement/question: Is E-M energy conserved locally? ... yes, but...

Consider an arrangement of charges & currents that generate $\vec{E}$ & $\vec{B}$ fields. Now let the charges & currents evolve under the influence of the $\vec{E}$ & $\vec{B}$ fields that they are generating!

$\longrightarrow$ we need to consider mechanical energy!

Equation of motion for free charges : $\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$

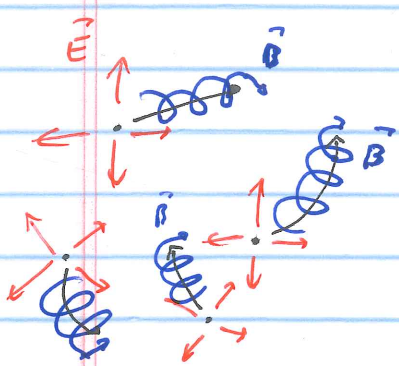
(currents are charges in motion) (Lorentz force law)

Mechanical work :

$$dW = d\,\text{work} = \vec{F}\cdot d\vec{l}$$
$$= q(\vec{E} + \vec{v}\times\vec{B})\cdot d\vec{l}$$
$$= q(\vec{E} + \underbrace{\cancel{\vec{v}\times\vec{B})\cdot\vec{v}}}_{\vec{B}\text{-fields can do no work}}\,dt \quad 0$$
$$= q\,\vec{E}\cdot\vec{v}\,dt$$
$$= \vec{E}\cdot(q\vec{v})\,dt$$

generalize

$$= \vec{E}\cdot\underbrace{\rho\vec{v}}_{\vec{J}}\,d^3r\,dt$$
$$= \vec{E}\cdot\vec{J}\,d^3r\,dt$$



Thus $\frac{dW}{dt} = \vec{E}\cdot\vec{J}\,d^3r$ in an infinitesimal volume

↳ in a finite volume V: $\left.\frac{dW}{dt}\right|_V = \int_V \vec{E}\cdot\vec{J}\,d^3r$

Action of fields on charges = E-M sourced mechanical power delivered in volume V. (instantaneous)

Short term objective reduce $\frac{dW}{dt}$ to a function of $\vec{E}$ & $\vec{B}$ (no $\vec{J}$, no q)

Ampère's improved law states :

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \iff \vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

thus $\vec{E}\cdot\vec{J} = \frac{1}{\mu_0}\underbrace{\vec{E}\cdot(\vec{\nabla}\times\vec{B})}_{\vec{B}\cdot\underbrace{(\vec{\nabla}\times\vec{E})}_{-\frac{\partial\vec{B}}{\partial t}} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})} - \varepsilon_0\,\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$

(Faraday's law)

$\Rightarrow \vec{E}\cdot\vec{J} = -\frac{1}{\mu_0}\underbrace{\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}}_{\frac{1}{2}\frac{\partial}{\partial t}(\vec{B})^2} - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B}) - \varepsilon_0\underbrace{\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}}_{\frac{1}{2}\frac{\partial}{\partial t}(\vec{E})^2}$

$\Rightarrow \vec{E}\cdot\vec{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left[\varepsilon_0\vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right] - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})$

so we have then

$$\frac{dW}{dt} = \int_V \vec{E}\cdot\vec{J}\,d^3r$$

$$= -\frac{\partial}{\partial t}\int_V \frac{1}{2}\left(\varepsilon_0\vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right)d^3r - \frac{1}{\mu_0}\underbrace{\int_V \vec{\nabla}\cdot(\vec{E}\times\vec{B})\,d^3r}$$

divergence theorem: $\oint_{\text{bounding surface of } V} (\vec{E}\times\vec{B})\cdot d\vec{s}$

Poynting's theorem:

$$\left.\frac{dW}{dt}\right|_V = -\frac{\partial}{\partial t}\underbrace{\left\{\int_V \frac{1}{2}\left(\varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right) d^3r\right\}}_{\substack{\text{Energy stored in} \\ \text{E-M field} \\ = E_{EM}|_V}} - \oint_{\substack{\text{bounding} \\ \text{surface} \\ \text{of } V}} \underbrace{\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right)}_{\substack{\vec{S} = \text{poynting} \\ \text{vector} \\ = \frac{\vec{E}\times\vec{B}}{\mu_0}}} \cdot d\vec{s}$$

Thus $\left.\frac{dW}{dt}\right|_V$ = the instantaneous change in E-M energy − Power flow in/out of volume V.

$$= -\frac{\partial}{\partial t} E_{EM}\Big|_V - \oint_{S(V)} \vec{S}\cdot d\vec{s}$$

<u>This is remarkable:</u> the gain/loss in mechanical energy due to work done on/by the charges is equal to the decrease/increase in stored energy of the E-M fields plus the energy that leaks ~~out/in~~ out of/into the volume of interest through the surface of V.

$$\boxed{\begin{aligned} \vec{S} &= \text{Energy per unit time per unit area} \\ &= \text{Power per unit area} \\ &= \text{``intensity''} \\[1em] \vec{S}\cdot d\vec{s} &= \text{Energy flux / ``current'' through a surface } d\vec{s} \end{aligned}}$$

Work done on charges becomes mechanical energy

$$\frac{dW}{dt} = \frac{\partial}{\partial t} E_{mech}\Big|_V = -\frac{\partial}{\partial t} E_{EM}\Big|_V - \oint_{S(V)} \vec{S}\cdot d\vec{s}$$

⚠ derivation ignores possibility that energy can leave V through exit of charges/currents

$$\Leftrightarrow \frac{\partial}{\partial t}\left(E_{Mech} + E_{EM}\right)\Big|_V = \underbrace{-\oint_{S(V)} \vec{S}\cdot d\vec{s}}_{\int_V (\vec{\nabla}\cdot\vec{S})\, d^3r}$$

~~Be~~ We consider energy densities: $\mathcal{E}_{mech}$ & $\mathcal{E}_{EM}$,

$$E_{Mech} = \int_V \mathcal{E}_{mech}(\vec{r})\, d^3r \quad \text{and} \quad E_M = \int_V \mathcal{E}_{EM}\, d^3r$$

$$= \int_V \underbrace{\frac{1}{2}\left(\varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right)}_{\mathcal{E}_{EM}} d^3r$$

Now we can write a continuity equation:

$$\boxed{\frac{\partial}{\partial t}\left(\mathcal{E}_{mech} + \mathcal{E}_{EM}\right) = -\vec{\nabla}\cdot\vec{S}}$$

local conservation of EM + mechanical Energy

since the equation is true for all volumes V, in particular infinitesimal ones.

take home message: EM work/energy can flow out (in) of a volume as EM work is done by the fields locally.

described by $\vec{S}$