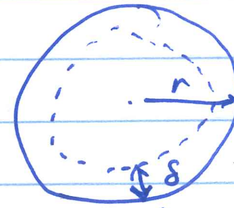
Tuesday, September 26, 2017

AC Skin Effect slides & demo



δ = skin depth

Electromagnetic Waves in Vacuum

Consider Maxwell's equations in vacuum with no charges and no currents [note the symmetry of $\vec{E}$ & $\vec{B}$ equations]

1) $\vec{\nabla}\cdot\vec{E}$

2) $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$

3) $\vec{\nabla}\cdot\vec{B} = 0$

4) $\vec{\nabla}\times\vec{B} = \varepsilon_0\mu_0\dfrac{\partial\vec{E}}{\partial t}$

Let's try to solve for $\vec{E}$ & $\vec{B}$

$$\vec{\nabla}\times(2) \Rightarrow \underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{E})}_{\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0\ (\text{eq. }1)}) - \vec{\nabla}^2\vec{E}} = -\frac{\partial}{\partial t}\underbrace{\vec{\nabla}\times\vec{B}}_{\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\ (\text{eq. }4)}$$

$$\Rightarrow -\vec{\nabla}^2\vec{E} = -\frac{\partial^2}{\partial t^2}(\mu_0\varepsilon_0\vec{E})$$

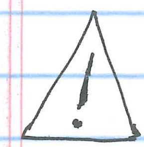
$$\Leftrightarrow \boxed{\vec{\nabla}^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\vec{E}} \quad (a)$$

Similarly,

$$\boxed{\vec{\nabla}^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\vec{B}} \quad (b)$$

For the E-field, we have
$$\begin{cases} \nabla^2 E_x = \mu_0 \varepsilon_0 \dfrac{\partial^2}{\partial t^2} E_x \\ \nabla^2 E_y = \mu_0 \varepsilon_0 \dfrac{\partial^2}{\partial t^2} E_y \\ \nabla^2 E_z = \mu_0 \varepsilon_0 \dfrac{\partial^2}{\partial t^2} E_z \end{cases}$$

In this form E_x, E_y, E_z are "independent" of each other and of $\vec{B}$!



Any solution of Maxwell's equations (in vacuum) satisfies (a) & (b) equations, but the converse is not necessarily true.

Monochromatic planewave solution

Consider the travelling wave solution: $\vec{E}(\vec{r},t) = E_0 \cos(kz - \omega t)\,\hat{x}$

$\sim E_0 e^{i(kz-\omega t)}\,\hat{x}$

(comment on AC skin effect problem)

insert into (a): $\nabla^2 E_x(\vec{r},t) = \mu_0 \varepsilon_0 \dfrac{\partial^2}{\partial t^2} E_x(\vec{r},t)$

$$\Leftrightarrow \cancel{\frac{\partial^2}{\partial x^2} E_0 \cos(kz-\omega t)} + \cancel{\frac{\partial^2}{\partial y^2} E_0 \cos(kz-\omega t)} + \underbrace{\frac{\partial^2}{\partial z^2} E_0 \cos(kz-\omega t)}_{-E_0 k^2 \cos(kz-\omega t)}$$

$$= \mu_0 \varepsilon_0 \underbrace{\frac{\partial^2}{\partial t^2} E_0 \cos(kz-\omega t)}_{-E_0 \omega^2 \cos(kz-\omega t)}$$

$$\Rightarrow -\cancel{E_0} k^2 \cancel{\cos(kz-\omega t)} = \mu_0 \varepsilon_0 (-\cancel{E_0}) \omega^2 \cancel{\cos(kz-\omega t)}$$

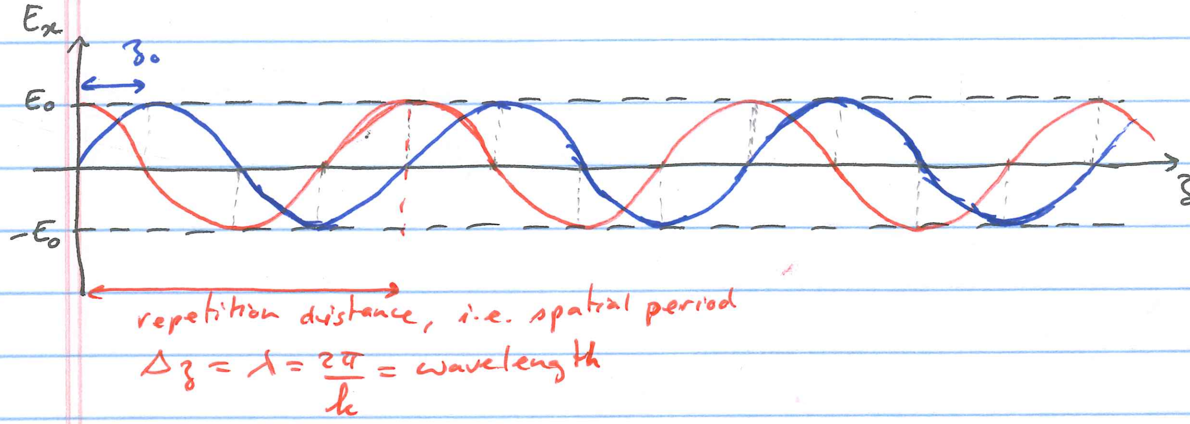
$$\Rightarrow \quad k^2 = \mu_0 \varepsilon_0 \omega^2$$

$\Rightarrow$ the plane wave satisfies equation (a) only if $\dfrac{\omega^2}{k^2} = \dfrac{1}{\mu_0 \varepsilon_0}$

Physics of the plane wave

Consider the planewave at $t=0$: $\vec{E} = E_0 \cos(kz)\,\hat{x}$

$$\Rightarrow E_x = E_0 \cos(kz)$$



Consider a function $f(z)$, then $f(z-z_0)$ is $f(z)$ but with the origin shifted to z_0.

Consider $E_x(z, t=0)$, then $E_x(\underbrace{z - \tfrac{\omega t}{k}}_{z_0}) = E_0 \cos[\underbrace{k(z - \tfrac{\omega t}{k})}_{kz-\omega t}]$
$= E_0 \cos(kz)$
$= E_x(z)$

$\Rightarrow$ you can think of the planewave as a spatial sinusoid that <u>translates in time</u> with $z_0 = vt = \dfrac{\omega}{k} t$

$\Rightarrow v = \dfrac{\omega}{k} = c$ is the speed of translation (i.e. speed of the wave)

$\boxed{\begin{array}{l}\text{Q: what's "plane" or "planar" about a plane wave?}\\ \text{A: Any } x\text{-}y \text{ plane is a plane of constant phase.}\end{array}}$

the wave has a $\begin{cases} \text{temporal frequency: } \omega \rightarrow \text{rads/s} \\ \text{spatial frequency: } k \rightarrow \text{rads/m} \end{cases}$

but $\frac{\omega^2}{k^2} = \frac{1}{\mu_0 \varepsilon_0} \Rightarrow$ $\boxed{\text{speed of wave (light)} = v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c}$

$\boxed{\text{Speed of EM wave (light) is constant regardless of } \underline{k} \text{ or } \underline{\omega}}$

$\lambda \quad f$

$k = \frac{2\pi}{\lambda} \qquad \omega = 2\pi f = \frac{2\pi}{T}$

not true for EM waves in matter (still mostly true)

also ~~not~~ true for sound waves, but not true for deBroglie waves

$mv = p = \hbar k \Rightarrow v = \hbar k / m$

$\frac{1}{2} m v^2 = E = \hbar \omega \Rightarrow v = \sqrt{\frac{2\hbar\omega}{m}}$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \left(\frac{\cancel{N}}{A^2}\right)_{\cancel{C^2}/s^2} \; 8.85 \times 10^{-12} \left(\frac{\cancel{C^2}}{\cancel{N} m^2}\right)}}$$

$= 2.99792 \times 10^8$ m/s $\simeq 3 \times 10^8$ m/s = speed of light

(300,000 km/s)

$\simeq 1$ ft/ns

Ex 1: Earth-Moon distance $\simeq 3.8 \times 10^8$ m

$\Rightarrow$ it takes EM waves ~ 1.25 s to cover this distance (Apollo ~ 4 days)

Ex 2: c is not that fast either

Round trip for light from Williamsburg-to-DC and back is about 1 ms ... almost an eternity for PHYS 252 planewave: $\vec{E} = E_0 \cos(kz - \omega t)\hat{x}$ (solution of $\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}$)

Q: Does planewave satisfy Maxwell's equations?

1) $\vec{\nabla}\cdot\vec{E} \overset{?}{=} 0 \iff \cancel{\frac{\partial E_x}{\partial x}}^{0} + \cancel{\frac{\partial E_y}{\partial y}}^{0} + \cancel{\frac{\partial E_z}{\partial z}}^{0} \overset{?}{=} 0$ ✓

⚠ Note that $\vec{E} = E_0 \cos(kx - \omega t)\hat{x}$ satisfies the wave equation, but not $\vec{\nabla}\cdot\vec{E} = 0$

$\Rightarrow$ the planewave must be transverse.

i.e. $\vec{E}$ (and $\vec{B}$) must be perpendicular to the direction of propagation.

stopped here

2) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$(\partial_x, \partial_y, \partial_z) \times (E_x, E_y, E_z) = (\cancel{\partial_y E_z}^{0} - \cancel{\partial_z E_y}^{0}, \partial_z E_x - \cancel{\partial_x E_z}^{0}, \cancel{\partial_x E_y}^{0} - \cancel{\partial_y E_x}^{0}) = -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \underbrace{\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}}, \quad 0 = \frac{\partial B_x}{\partial t}, \quad 0 = \frac{\partial B_z}{\partial t}$$

$\to B_x = cst$ $\quad$ $\to B_z = cst$ — not wave like

$$-E_0 k \sin(kz - \omega t) = -\frac{\partial}{\partial t} B_y$$

$$\Rightarrow B_y = E_0 \frac{k}{\omega}\cos(kz - \omega t) + cst$$

$\Rightarrow$ Maxwell's equations (Faraday's law) require an accompanying B-field planewave (exactly in phase):

$$\boxed{\vec{B} = E_0 \frac{k}{\omega}\cos(kz - \omega t)\hat{y} = \frac{E_0}{c}\cos(kz - \omega t)\hat{y} = B_0 \cos(kz - \omega t)\hat{y} \quad \text{with } B_0 = E_0/c}$$