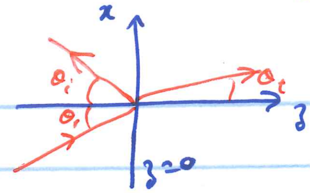
quiz: class started with boundary condition quiz

Tuesday, October 10, 2017



The overall form of the boundary conditions can be simplified, since at the interface ($z=0$), all of the exponents are equal (for all t & (x,y), with $z=0$)

$$[?]\,\cancel{e^{i(\vec{k}_i\cdot\vec{r}-\omega_i t)}} + [??]\,\cancel{e^{i(\vec{k}_r\cdot\vec{r}-\omega_r t)}} = [???]\,\cancel{e^{i(\vec{k}_t\cdot\vec{r}-\omega_t t)}}$$

(for $z=0$)

The specific boundary conditions become:

1) $\varepsilon_1 E_{1,\perp} = \varepsilon_2 E_{2,\perp} \Rightarrow \varepsilon_1 (E_{0,z} + E_{0,r,z}) = \varepsilon_2 E_{0,t,z}$

2) $\vec{E}_{1,\parallel} = \vec{E}_{2,\parallel} \Rightarrow (E_0 + E_{0,r})_{x,y} = (E_{0,t})_{x,y}$ 2 equations

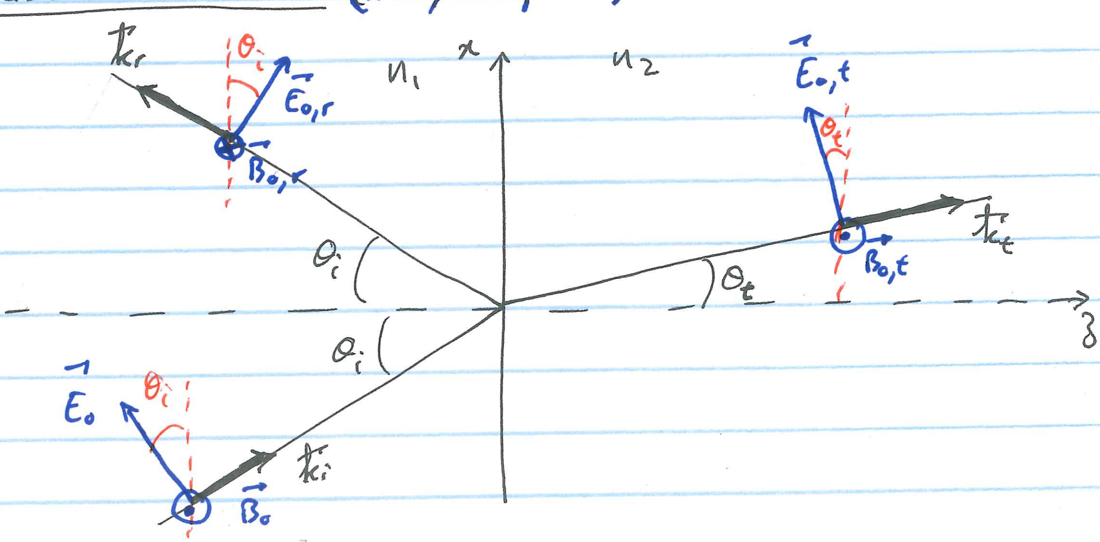
3) $B_{1,\perp} = B_{2,\perp} \Rightarrow B_{0,z} + B_{0,r,z} = B_{0,t,z}$

4) $\dfrac{\vec{B}_{1,\parallel}}{\mu_1} = \dfrac{\vec{B}_{2,\parallel}}{\mu_2} \Rightarrow \dfrac{1}{\mu_1}(B_0 + B_{0,r})_{x,y} = \dfrac{1}{\mu_2}(B_{0,t})_{x,y}$ 2 equations

also 5) $|\vec{B}_0| = \dfrac{n|\vec{E}_0|}{c}$ and $\hat{E}_0 \times \hat{B}_0 = \hat{k}$

P-polarized Incident EM waves (no y-component)

S-polarized has the same physics as normal incidence but with more trigonometry
[see problem 9.17.



#2

b.c. 1) $\varepsilon_1 \left(-E_0 \sin\theta_i + E_{0,r} \sin\theta_i\right) = -\varepsilon_2 E_{0,t} \sin\theta_t$

$\times\ \mu_{1,2} \simeq \mu_0 \Rightarrow$ $\Rightarrow \mu\varepsilon = \frac{1}{c_m^2} = \frac{n^2}{c^2}$ $\quad \hookrightarrow n_1^2\left(-E_0 \sin\theta_i + E_{0,r} \sin\theta_i\right) = -n_2^2 E_{0,t} \sin\theta_t$

b.c. 2) $E_0 \cos\theta_i + E_{0,r} \cos\theta_i = E_{0,t} \cos\theta_t$

b.c. 3) $0 + 0 = 0$

b.c. 4) $\frac{1}{\mu_1}\left(B_0 - B_{0,r}\right) = \frac{1}{\mu_2} B_{0,t}$

assume

$\mu_1 \simeq \mu_2 \simeq \mu_0$

$B_0 - B_{0,r} = B_{0,t} \Rightarrow \frac{n_1 E_0}{c} - \frac{n_1}{c} E_{0,r} \simeq \frac{n_2}{c} E_{0,t}$

$\Rightarrow E_0 - E_{0,r} \simeq \frac{n_2}{n_1} E_{0,t}$

$n_i \sin\theta_i = n_t \sin\theta_t$ (Snell's law)

b.c. 1) $\Rightarrow n_1^{\cancel{2}}\left(-E_0 + E_{0,r}\right) \cancel{\sin\theta_i} = -n_2^{\cancel{2}} E_{0,t} \cancel{\sin\theta_t}$

$\Rightarrow E_0 - E_{0,r} = \frac{n_2}{n_1} E_{0,t}$ same as b.c. 4)

b.c. 2) $\Rightarrow E_0 + E_{0,r} = E_{0,t} \frac{\cos\theta_t}{\cos\theta_i}$

Set $\left\{ \begin{array}{l} E_{0,r} = r E_0 \\ E_{0,t} = t E_0 \end{array} \right\} \Rightarrow$ $\boxed{\left\{ \begin{array}{ll} 1 - r = \frac{n_2}{n_1} t & (1) \\ 1 + r = \frac{\cos\theta_t}{\cos\theta_i} t & (2) \end{array} \right.}$

(consistent with normal incidence case)

$(1) + (2) \Rightarrow 2 = \left(\frac{n_2}{n_1} + \frac{\cos\theta_t}{\cos\theta_i}\right) t$

$$\Longleftrightarrow \quad \boxed{t = \frac{2}{\frac{n_2}{n_1} + \frac{\cos\theta_t}{\cos\theta_i}}}$$

plug "t" into (2)

$$r = \frac{\cos\theta_t}{\cos\theta_i}\left(\frac{2}{\frac{n_2}{n_1} + \frac{\cos\theta_t}{\cos\theta_i}}\right) - 1$$

$$\Longleftrightarrow \quad \boxed{r = \frac{\frac{\cos\theta_t}{\cos\theta_i} - \frac{n_2}{n_1}}{\frac{\cos\theta_t}{\cos\theta_i} + \frac{n_2}{n_1}}}$$

Fresnel's Equations

note that $\cos\theta_t = \sqrt{1 - \sin^2\theta_t} \xrightarrow{\text{Snell's law}} = \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i}$

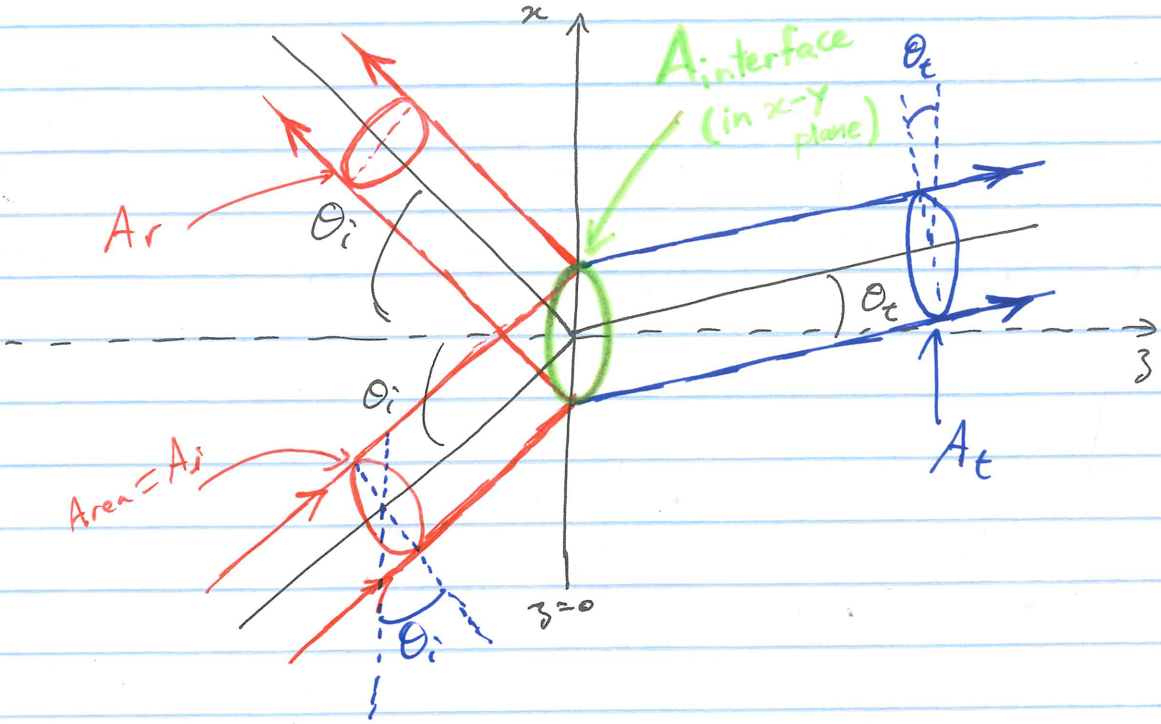
$$\boxed{\begin{aligned} t &= \frac{2}{\frac{\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i}}{\cos\theta_i} + \frac{n_2}{n_1}} \\ r &= \frac{\frac{\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i}}{\cos\theta_i} - \frac{n_2}{n_1}}{\frac{\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i}}{\cos\theta_i} + \frac{n_2}{n_1}} \end{aligned}}$$

Fresnel's Equations

Show overheads on Fresnel Equations w/ $\left\{\begin{matrix}\text{transmission}\\ \text{reflection}\end{matrix}\right\}$ amplitudes for glass

Transmission & Reflection Coefficients

Let's look at intensities:



Beam cross-sectional Areas:

$A_r = A_i$ $\qquad\qquad$ $A_{interface} \cos\theta_i = A_i = A_r$

but $A_{interface} \cos\theta_t = A_t$

$$\Rightarrow \quad \frac{A_t}{A_i} = \frac{\cos\theta_t}{\cos\theta_i}$$

thus R = reflection coefficient $= \dfrac{P_r}{P_i} = \dfrac{I_r \cancel{A_r}}{I_i \cancel{A_i}} = \dfrac{I_r}{I_i} = \dfrac{\langle|\vec{S}_r|\rangle}{\langle|\vec{S}_i|\rangle} = \dfrac{|\vec{E}_{o,r}|^2}{|E_o|^2} = r^2$

Power

T = transmission coefficient $= \dfrac{P_t}{P_i} = \dfrac{I_t A_t}{I_i A_i} = \dfrac{\overset{\frac{1}{2} n_2 c \varepsilon_0 E_{o,t}^2}{I_t}}{\underset{\frac{1}{2} n_1 c \varepsilon_0 E_o^2}{I_i}} \dfrac{\cos\theta_t}{\cos\theta_i} = \dfrac{\cos\theta_t}{\cos\theta_i} \dfrac{n_2}{n_1} t^2$

thus $\boxed{\begin{aligned} R &= r^2 \\ T &= \frac{\cos\theta_t}{\cos\theta_i}\,\frac{n_2}{n_1}\,t^2 \end{aligned}}$

Show Transmission and Reflection coefficients for glass and silicon.

$\hookrightarrow$ mention glancing angle reflection for x-rays

Brewster's Angle

1) Mathematical Approach:

Brewster condition $\Leftrightarrow$ $r = 0$

$$\frac{\dfrac{\sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B}}{\cos\theta_B} - \dfrac{n_2}{n_1}}{\dfrac{\sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B}}{\cos\theta_B} + \dfrac{n_2}{n_1}} = 0$$

$\Rightarrow$ numerator $= 0$ $\Leftrightarrow$ $\left(\dfrac{\sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B}}{\cos\theta_B}\right)^2 = \left(\dfrac{n_2}{n_1}\right)^2$

$$\Leftrightarrow \quad 1-\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B = \left(\frac{n_2}{n_1}\right)^2 \underbrace{\cos^2\theta_B}_{1-\sin^2\theta_B}$$

$$\Leftrightarrow \quad 1-\left(\frac{n_2}{n_1}\right)^2 = \left[\left(\frac{n_1}{n_2}\right)^2 - \left(\frac{n_2}{n_1}\right)^2\right]\sin^2\theta_B$$

$$\Leftrightarrow \frac{\left(\frac{n_2}{n_1}\right)^2 \times \quad 1-\left(\frac{n_2}{n_1}\right)^2}{\left(\frac{n_2}{n_1}\right)^2 \times \left(\frac{n_1}{n_2}\right)^2 - \left(\frac{n_2}{n_1}\right)^2} = \sin^2\theta_B$$

$$\Rightarrow \frac{\left(\frac{n_2}{n_1}\right)^2 \cancel{\left(1-\left(\frac{n_2}{n_1}\right)^2\right)}}{\underbrace{1-\left(\frac{n_2}{n_1}\right)^4}_{\cancel{\left[1-\left(\frac{n_2}{n_1}\right)^2\right]}\left[1+\left(\frac{n_2}{n_1}\right)^2\right]}} = \sin^2\theta_B$$

$$\Rightarrow \sin^2\theta_B = \frac{\left(\frac{n_2}{n_1}\right)^2}{1+\left(\frac{n_2}{n_1}\right)^2} \quad \Rightarrow \cos^2\theta_B = 1-\sin^2\theta_B = \frac{1}{1+\left(\frac{n_2}{n_1}\right)^2}$$

$$\Rightarrow \tan^2\theta_B = \frac{\sin^2\theta_B}{\cos^2\theta_B} = \left(\frac{n_2}{n_1}\right)^2 \Rightarrow \boxed{\tan\theta_B = \frac{n_2}{n_1}}$$

for $n_2 = 1.5$ glass

$n_1 = 1.0$ air $\Rightarrow \boxed{\theta_B = 56°}$

you can use Brewster's angle to measure index of refraction (n_2)