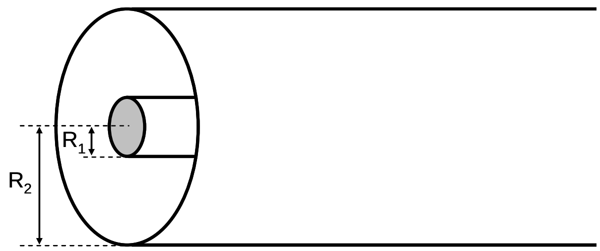
PHYS 402: Electricity & Magnetism II
Due date: Thursday, November 2, 2017

Problem Set #6: Electronics 252 revisited ...

I. LC model of a coaxial cable

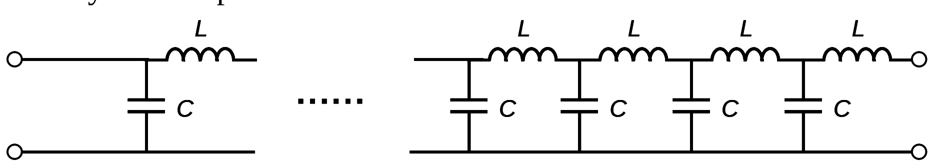
Consider a coaxial metal cable consisting of an inner full metal cylinder of radius R_1 and an outer metal cylindrical shell of radius R_2, as shown in the figure below.



1. Calculate the capacitance per unit length of the coaxial cable (neglect any edge effects).

2. Calculate the inductance per unit length if the inner and outer conductors are attached on one end by a thin wire (neglect any edge effects).

3. We want to derive an expression for the impedance Z_{coax}, or effective resistance, of the coaxial cable. We model the coaxial cable as the following infinite ladder circuit of infinitely small capacitors and inductors:



a. If you add a capacitor and inductor pair to the infinite coaxial ladder circuit, then the impedance of the total circuit should not change. Use this principle to show that the impdedance, Z_{coax}, of the infinite ladder circuit of capacitors and inductors satisfies the relation:

$$Z_{coax} = \frac{(Z_L + Z_{coax})Z_C}{(Z_L + Z_{coax}) + Z_C}$$

where $Z_L = i\omega L$ is the impedance of the inductors, $Z_L = 1/i\omega C$ is the impedance of the capacitors, ω is the frequency of the current running through the circuit, and $\boldsymbol{i}$ the imaginary number.

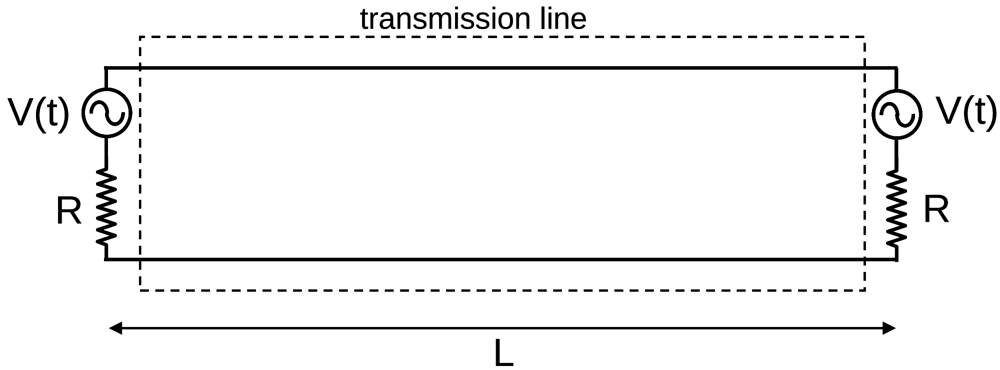
b. Derive an expression for Z_{coax} as function of L, C, and ω. If L and C are the inductance and capacitance for a short piece of cable $\boldsymbol{d}$, determine an expression for Z_{coax} as $d \to 0$.

c. Compare your result with the expression derived in class for Z_{coax}.

II. Nyquist's derivation of Johnson Noise

Thermal agitation of electrons produces the resistance to electrical conduction in a resistor with resistance R. This same thermal agitation moves the electrons around randomly inside the resistor so as to produce a small fluctuating voltage, V(t), across the resistor terminals, which is referred to as Johnson noise. Since R and V(t) are produced by the same phenomena (thermal agitation of electrons), they are also related. In this problem, you will derive the relation between V(t), R, and T (temperature).

We will model a resistor with a fluctuating voltage, V(t), across its terminals as an ideal resistor in series with a fluctuating signal generator. In Nyquist's derivation, two identical "noisy" resistors with resistance R are connected via a transmission line with impedance R, so as to produce a 1-d electrical circuit equivalent of the blackbody radiation problem, as shown in the figure below. The impedance R of the transmission line guarantees that signals generated on one end of the transmission line will be completely absorbed at the other end, without reflections.



1. Transmission line modes

The boundary condition for the transmission line is that the electromagnetic field must have a node at either end of the transmission line. The speed of light in the transmission line is c_t.

a. Calculate the permitted wavelengths of the modes of the electromagnetic field of the transmission line.

b. Calculate the permitted frequencies of the modes of the electromagnetic field of the transmission line.

c. Show that over a large frequency span, Δf, the number of modes of the electromagnetic field is $N(\Delta f) = (2L/c_t)\Delta f$.

2. Thermal population of the electromagnetic modes

According to the equipartition theorem, each mode of the electromagnetic field (i.e. degree of freedom) has a total energy of kT stored in it, where k is Boltzmann's constant. This energy comes from the two resistors which are both at temperature T.

a. How long does it take for thermal energy emitted by one resistor to arrive at the other resistor (and be absorbed)?

b. Show that the electromagnetic power $dP(f)$, in a frequency band df, absorbed by a resistor is $dP(f)=kTdf$.

3. Johnson noise

In thermal equilibrium, the power absorbed by a resistor in a given frequency range is also the power emitted by the resistor in the same frequency range due to the fluctuating voltage, *V(t)*, on its terminals.

a. Calculate the current *I* generated by the fluctuating voltage source on one of the two resistors.

b. Calculate the electrical power dissipated in one resistor due to the current generated by the fluctuating voltage source of the other resistor.

c. Show that over a frequency range Δf, the RMS value of the fluctuating voltage (i.e. Johnson noise) from a single resistor must be given by the expression:

$$\langle V_{RMS} \rangle = \sqrt{4RkT\Delta f}$$

d. Calculate the RMS Johnson voltage noise for a 10 MΩ resistor at room temperature over a 1 kHz bandwidth.

III. Reflection coefficient for an arbitrarily terminated transmission line

Consider a transmission line with complex impedance Z_C terminated by a complex load Z_L. A TEM wave in the transmission line (voltage and current waves related by Z_c) is incident on the load and produces a voltage and current in the load (related by Z_L), and also produces a reflected TEM wave (voltage and current waves related by Z_c).

a. Show or explain with the help of a circuit diagram that

$$V_{load}=V_{incident}+V_{reflected}$$

$$I_{load}=I_{incident}-I_{reflected}$$

You may assume that the region where the transmission line is terminated by the load is much smaller than the wavelength of the TEM wave, so that Kirchhoff laws apply.

b. Show that the voltage reflection coefficient is given by

$$\Gamma = \frac{V_{reflected}}{V_{incident}} = \frac{Z_L - Z_C}{Z_L + Z_C}$$

IV. TE and TM modes of a box cavity

Read Griffiths section 9.5 ("Guided Waves").

Do problem 9.40 in 4th Ed. [9.38 in 3rd Ed.].

V. EM plane wave in a conductor

Problem 9.20 in 4th Ed. [9.19 in 3rd Ed.].