Tuesday, October 24, 2017

EM wave equations in a conductor:

$$\boxed{\vec{\nabla}^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}}$$

$$\boxed{\vec{\nabla}^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu\sigma \frac{\partial \vec{B}}{\partial t}}$$

If we assume a plane wave solution (propagating in $+\hat{z}$ direction)

$\vec{E}(z,t) = \vec{E}_0 e^{i(kz-\omega t)}$ and $\vec{B}(z,t) = \vec{B}_0 e^{i(kz-\omega t)}$

then we find that this solution works, so long as

$$\boxed{k^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega}$$

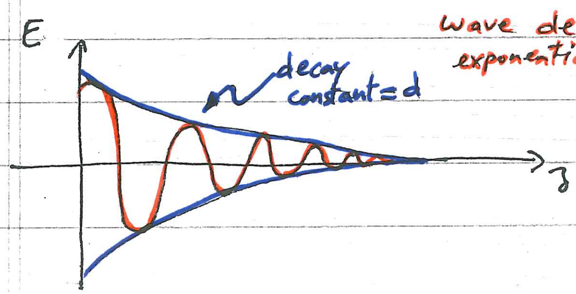
example of a dispersion relation.

other examples: $k = \frac{\omega}{c}$ (light in vacuum)

$k^2 = \frac{2m\omega}{\hbar}$

↳ k is complex: $k = k_0 + i\frac{1}{d}$

thus $\vec{E}(z,t) = \vec{E}_0 \underbrace{e^{-\frac{z}{d}}}_{\text{wave decays exponentially}} e^{i(k_0 z-\omega t)}$



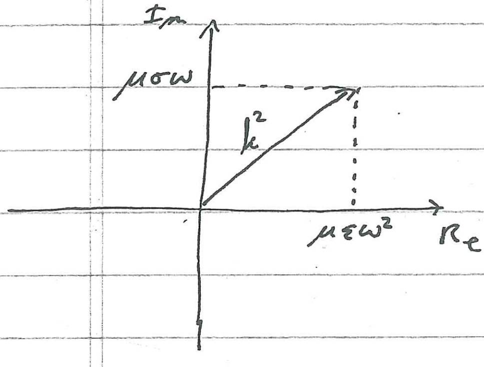
$\boxed{d > 0 \rightarrow \text{wave decays}}$

$d < 0 \rightarrow$ wave blows up

↳ energy cannot be conserved

determination of k_o & d

Let's put k^2 in polar form (easier to take $\sqrt{\ }$)



$$k^2 = |k^2| e^{i\theta}$$

$$\Rightarrow k = \sqrt{|k^2|}\, e^{i\frac{\theta}{2}} \pm \pi$$

$$= \sqrt{|k^2|}\left[\cos(\theta/2) + i \sin(\theta/2)\right]$$

k_o

$\frac{1}{d}$

$$k^2 = \mu\varepsilon\omega^2\left(1 + i\frac{\sigma}{\varepsilon\omega}\right)$$

$$\Rightarrow |k^2| = \mu\varepsilon\omega^2\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}$$

also
$$1 + \frac{i\sigma}{\varepsilon\omega} = \sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}\left[\underbrace{\frac{1}{\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}}}_{\cos\theta} + i\underbrace{\frac{\frac{\sigma}{\varepsilon\omega}}{\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}}}_{\sin\theta}\right]$$

thus $\cos\theta = \dfrac{1}{\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}}$ and $\sin\theta = \dfrac{\sigma/\varepsilon\omega}{\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}}$

$$\Rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sigma}{\varepsilon\omega}$$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2} + \frac{1}{2}\cos\theta \Rightarrow \cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}}\right)}$$

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos\theta \Rightarrow \sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{2}\left(1 - \frac{1}{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}}\right)}$$

thus,

$$\sqrt{1 + i\frac{\sigma}{\varepsilon\omega}} = \sqrt{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}}\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)$$

$$= \sqrt{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}}\left\{\sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}}\right)} + i\sqrt{\frac{1}{2}\left(1-\frac{1}{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}}\right)}\right\}$$

$$= \underbrace{\overset{\sqrt{\mu\varepsilon\omega^2}}{\sqrt{\frac{1}{2}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right)}}}_{k_0} + i\underbrace{\overset{\sqrt{\mu\varepsilon\omega^2}}{\sqrt{\frac{1}{2}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right)}}}_{1/d}$$

$$\text{thus } k_0 = \omega\sqrt{\frac{\mu\varepsilon}{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1} = \underbrace{\frac{\omega}{c}}_{k_n} n \frac{1}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1}$$

$$\Rightarrow k_0 = k_{vacuum}\, n(\omega)$$

frequency dependent index of refraction (dispersion)

$$\frac{1}{d} = \omega\sqrt{\frac{\mu\varepsilon}{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1}$$

for copper: $\sigma_{copper} \simeq 5.7\times10^{7}\ (\Omega\cdot m)^{-1}$

$$\varepsilon \sim \varepsilon_0 = 8.85\times10^{-12}\ \frac{C^2}{Nm^2}$$

$$\Rightarrow \frac{\sigma}{\varepsilon\omega} \sim \frac{10^8}{10^{-11}\omega} \simeq \frac{10^{19}}{\omega} \gg 1 \quad \text{for} \quad 0 < \omega < 10^{15}$$

optical

[$\omega \sim 10^{19} \rightarrow$ soft x-rays] so $\frac{1}{d} \simeq \sqrt{\omega}\sqrt{\frac{\mu \cancel{\varepsilon}}{2}}\sqrt{\frac{\sigma}{\cancel{\varepsilon}\omega}} \Rightarrow d = \sqrt{\frac{2}{\mu\sigma\omega}}$

note : σ = conductivity
ρ = resistivity $= \frac{1}{\sigma}$ $\Rightarrow$ $\boxed{d = \sqrt{\frac{2\rho}{\mu\omega}} = \text{skin depth}}$

$\Rightarrow$ EM wave propagates a distance d in metal before being absorbed (E is reduced by ~3)

Some numbers for good conductors (Cu, Ag, Al, ...)

$d(60\ Hz) \simeq 1\ cm$
$d(1\ kHz) \simeq 2\ mm$
$d(100\ kHz) \simeq 200\ \mu m$
$d(1\ MHz) \simeq 60\ \mu m$
$d(1\ GHz) \simeq 2\ \mu m$

$\Rightarrow$ it is relatively easy to shield RF signals
(also, RF circuits tend to have very skinny wires)

From Faraday's law, we get (if $\bar{E}_0 = E_0 \hat{x}$)

$$\bar{B}(z,t) = \frac{k}{\omega} E_0\, e^{-z/d}\, e^{i(k_0 z - \omega t)}\, \hat{y}$$

$$k = k_0 + \frac{i}{d} = |k|e^{i\phi} = |k|e^{i\theta/2}$$

$$|k| = \omega\sqrt{\mu\varepsilon}\sqrt{\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}} \quad \text{and} \quad \tan(\phi) = \tan(\theta/2) = \frac{b}{a} = \frac{1}{k_0 d}$$

$k = a + ib$

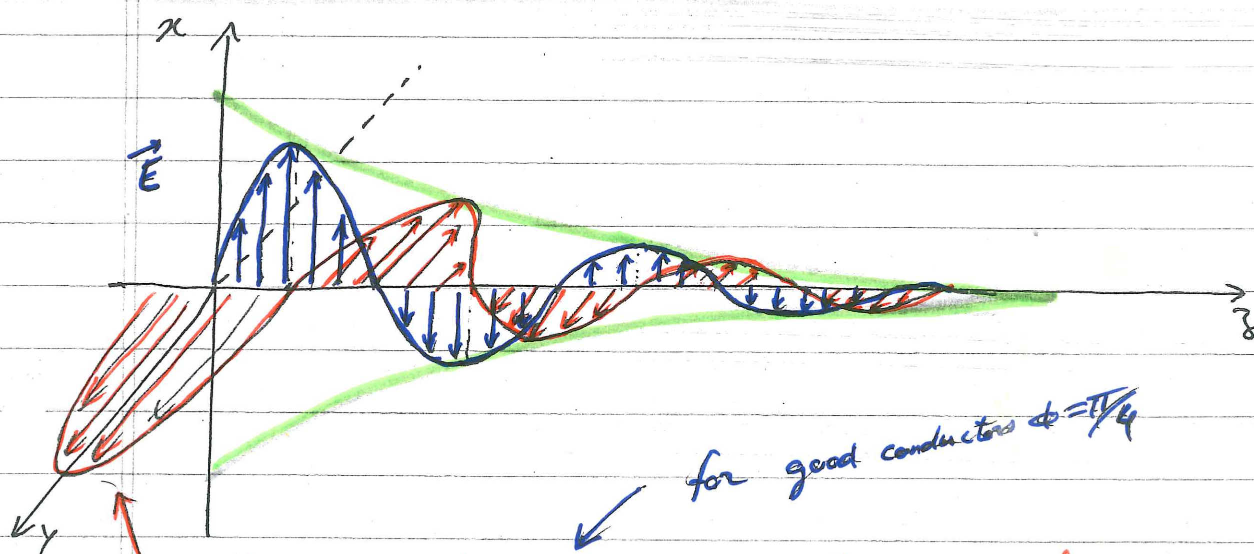
$$\text{thus } \vec{B}(z,t) = \frac{\cancel{\omega}\overbrace{\sqrt{\mu\varepsilon}}^{n/c}}{\cancel{\omega}} \underbrace{\left[1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{1/4}}_{\sim\sqrt{\frac{\sigma}{\varepsilon\omega}}} E_0\, e^{-z/d}\, e^{i(k_0 z-\omega t+\phi)}\, \hat{y}$$

$$\simeq \underbrace{\frac{n}{c}\sqrt{\frac{\sigma}{\varepsilon\omega}}}_{} E_0\, e^{-z/d}\, e^{i(k_0 z-\omega t+\phi)}\, \hat{y}$$

this term is large

↳ most of the EM energy is in B-field

note: $\vec{E}(z,t) = E_0\, e^{-z/d}\, e^{i(k_0 z-\omega t)}\, \hat{x}$



for good conductors $\phi = \pi/4$

not necessarily $\pi/2$ phase delay $\quad B_0 = \frac{k E_0}{\omega} \neq \frac{1}{c_m} E_0$

$$= \left(k_0 + i\frac{1}{d}\right)\frac{E_0}{\omega}$$

<u>where did the energy go?</u> → "resistive heating of medium"
(i.e. absorption)

<u>note</u>: absorption & dispersion are always present togethers
(Kramers–Kronig relations)

<u>where did the momentum go?</u> → wave pushes material
↳ most energy lost as heat

Reflection & Transmission at a conducting surface

Recall boundary conditions:

1) $\epsilon_1 E_{1,\perp} - \epsilon_2 E_{2,\perp} = \sigma_f = 0$ 2) $\vec{E}_{1,\parallel} = \vec{E}_{2,\parallel}$

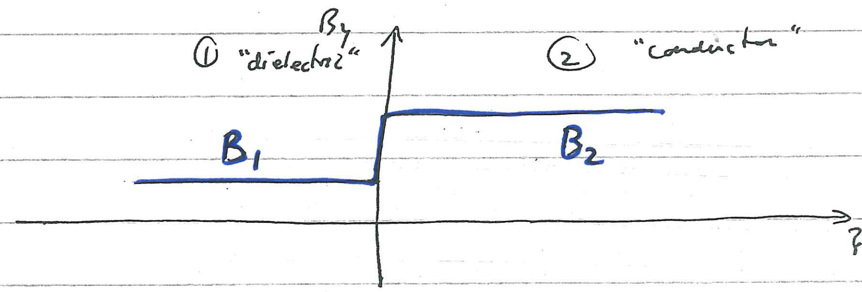
3) $B_{1,\perp} = B_{2,\perp}$ 4) $\frac{1}{\mu_1}\vec{B}_{1,\parallel} - \frac{1}{\mu_2}\vec{B}_{2,\parallel} = \vec{K}_f \times \hat{n}$

stopped here

$\boxed{\vec{K}_f = 0}$ (i.e. no free surface current ... however $\vec{J}_f \neq 0$)

↳ why? .... suppose $\mu_1 \simeq \mu_2 \simeq \mu_0$

if $\vec{K}_f \neq 0$, then $\vec{B}$ is discontinuous across the boundary



$$\Rightarrow \frac{\partial B_y}{\partial z} = \underbrace{\beta\,\delta(z)}_{\text{surface current}} \Rightarrow \vec{E}_{\text{surface}} = +\infty$$

(if you believe Ohm's law)

Ampère's law: $\vec{\nabla}\times\vec{B} = \mu\underbrace{\sigma\vec{E}}_{\vec{J}} + \mu\epsilon\frac{\partial \vec{E}}{\partial t}$ focus on this term leave out

$$\left(\cancelto{0}{\partial_y B_z} - \underbrace{\partial_z B_y}_{\beta\,\delta(z)},\ \cancelto{0}{\partial_z B_x} - \cancelto{0}{\partial_x B_z},\ \cancelto{0}{\partial_x B_y} - \cancelto{0}{\partial_y B_x}\right) = \left(\mu\sigma + \cancel{\mu\epsilon\frac{\partial}{\partial t}}\right)(E_x, E_y, E_z)$$

$\Rightarrow E_x = +\infty$ (assuming σ is finite)

$\Rightarrow$ also $\vec{K} \neq 0$ implies a surface $E_\parallel \rightarrow$ potential problem with b.c.