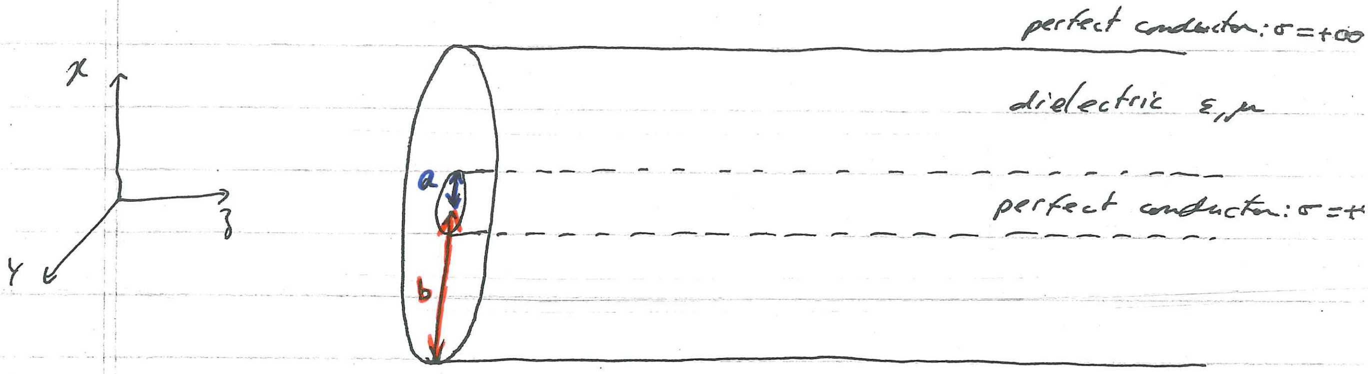
Thursday, November 2, 2017

Coaxial Cable Transmission Line

Consider a coaxial cable:



We will find the TEM (Transverse Electric - Magnetic) wave solutions (i.e. no longitudinal $\vec{E}$ or $\vec{B}$ fields)

$\Rightarrow$ assume:

$$\begin{cases} E_z = 0 \\ B_z = 0 \end{cases} \quad \Bigg| \quad \begin{matrix} k_x = 0 \\ k_y = 0 \end{matrix} \quad \Bigg| \quad k_z \neq 0$$

In the dielectric, the wave equation still holds so

$$\vec{E}_x = E_{0,x}(x,y)\, e^{i(kz-\omega t)}\, \hat{x}$$
$$\vec{E}_y = E_{0,y}(x,y)\, e^{i(kz-\omega t)}\, \hat{y}$$
$$\vec{B}_x = B_{0,x}(x,y)\, e^{i(kz-\omega t)}\, \hat{x}$$
$$\vec{B}_y = B_{0,y}(x,y)\, e^{i(kz-\omega t)}\, \hat{y}$$

We can put more constraints on $\vec{E}$ & $\vec{B}$ by applying Maxwell's equations in the dielectric ($\vec{J} = 0$).

1) $\vec{\nabla} \cdot \vec{E} = 0$ 2) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

3) $\vec{\nabla} \cdot \vec{B} = 0$ 4) $\vec{\nabla} \times \vec{B} = \varepsilon\mu \frac{\partial \vec{E}}{\partial t}$

1) $\Rightarrow \partial_x E_x + \partial_y E_y = 0 \Rightarrow$ $\boxed{\vec{\nabla}_{x,y} \cdot \vec{E}_{x,y} = 0 \quad \text{2D electrostatic condition}}$

2) $\Rightarrow \left(\cancelto{0}{\partial_y E_z} - \partial_z E_y,\ \partial_z E_x - \cancelto{0}{\partial_x E_z},\ \partial_x E_y - \partial_y E_x\right) = -\frac{\partial}{\partial t}\left(B_x, B_y, \cancelto{0}{B_z}\right)$

$\Rightarrow -\partial_z E_y = -\frac{\partial B_x}{\partial t} \Rightarrow -ik E_y = i\omega B_x \Rightarrow$ $\boxed{B_x = -\frac{k}{\omega} E_y = -\frac{n}{c} E_y}$

$\Rightarrow \partial_z E_x = -\partial_t B_y \Rightarrow ik E_x = i\omega B_y \Rightarrow$ $\boxed{B_y = \frac{k}{\omega} E_x = \frac{n}{c} E_x}$

$\Rightarrow \partial_x E_y - \partial_y E_x = 0 \Rightarrow$ $\boxed{\vec{\nabla}_{xy} \times \vec{E}_{xy} = 0 \quad \text{2D electrostatic condition}}$

3) $\Rightarrow \partial_x B_x + \partial_y B_y = 0 \Rightarrow$ $\boxed{\vec{\nabla}_{x,y} \cdot B_{x,y} = 0 \quad \text{2D magnetostatic condition}}$

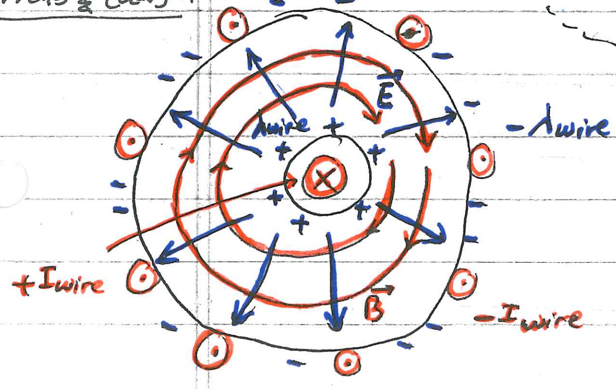
4) $\Rightarrow \left(\cancelto{0}{\partial_y B_z} - \partial_z B_y,\ \partial_z B_x - \cancelto{0}{\partial_x B_z},\ \partial_x B_y - \partial_y B_x\right) = \varepsilon\mu\, \partial_t\left(E_x, E_y, \cancelto{0}{E_z}\right)$

$\Rightarrow -ik B_y = \varepsilon\mu(-i\omega) E_x \Rightarrow$ $\boxed{B_y = \left(\frac{n}{c}\right)^2\left(\frac{\omega}{k}\right) E_x = \frac{n}{c} E_x}$

$\Rightarrow ik B_x = \varepsilon\mu(-i\omega) E_y \Rightarrow$ $\boxed{B_x = -\left(\frac{n}{c}\right)^2\left(\frac{\omega}{k}\right) E_y = -\frac{n}{c} E_y}$

$\Rightarrow \partial_x B_y - \partial_y B_x = 0 \Rightarrow$ $\boxed{\vec{\nabla}_{xy} \times \vec{B}_{xy} = 0 \quad \text{2D magnetostatic condition}}$

Currents & charges:



We can exploit the cylindrical symmetry of the cable:

In cylindrical coordinates, the solutions are:

Gauss's law: $2\pi r\, E_r = \frac{\lambda_{wire}}{\varepsilon} \Rightarrow E_r = \frac{\lambda_{wire}}{2\pi\varepsilon}\frac{1}{r}\hat{r}$

Ampère's law: $2\pi r\, B_\phi = \mu I_{wire} \Rightarrow B_\phi = \frac{\mu I_{wire}}{2\pi}\frac{1}{r}\hat{\phi}$

$\Rightarrow$ the solutions already satisfy the B.C. $\begin{cases} E_\parallel = 0 \\ B_\perp = 0 \end{cases}$

⚠ alternative solution: $B_r = \frac{\mu I_{wire}}{2\pi}\frac{1}{r}\hat{r}$; $E_\phi = \frac{\lambda_{wire}}{2\pi\varepsilon}\frac{1}{r}\hat{\phi}$

(i.e. satisfy $\vec{\nabla}\cdot\vec{E},\vec{B} = 0$, $\vec{\nabla}\times\vec{E},\vec{B} = 0$)

$\hookrightarrow$ these do not satisfy b.c. !!!

Since $\vec{\nabla}_{x,y}\times\vec{E} = 0$, there exists a potential in 2D (x,y) such that $\vec{E} = -\nabla V$

$\Rightarrow$ The concept of a voltage is applicable to TEM waves (and transmission lines)

$\hookrightarrow$ not obvious since "V" does not normally make sense in electrodynamics ($\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \neq 0$)

[for electrical engineering this is a big deal.]

$\hookrightarrow$ PHYS 252 is relevant for understanding transmission lines!

We need to solve $\nabla^2 V = 0$ for $a < r < b$

$$\Leftrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2}}}_{0 \text{ by symmetry}} = 0$$

$$\Leftrightarrow \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \Rightarrow r\frac{\partial V}{\partial r} = \underbrace{c_1}_{\text{constant } c_1} \Leftrightarrow \frac{\partial V}{\partial r} = \frac{c_1}{r}$$

$$\Rightarrow V = c_1 \ln r + c_2$$

"arbitrarily" define $\begin{cases} \text{inner conductor as having } V = V_0 \\ \text{outer conductor as having } V = 0 \text{ (ground)} \end{cases}$

thus $\begin{cases} V_0 = c_1 \ln a + c_2 & (i) \\ 0 = c_1 \ln b + c_2 & (ii) \end{cases}$

$(i) - (ii) \Rightarrow V_0 = c_1 \ln a - c_1 \ln b$

$\Leftrightarrow V_0 = c_1 \ln \frac{a}{b} \Leftrightarrow c_1 = \frac{V_0}{\ln(a/b)}$

$\hookrightarrow$ plug into (ii) : $0 = \frac{V_0}{\ln(a/b)} \ln b + c_2 \Rightarrow c_2 = \frac{-V_0 \ln(b)}{\ln(a/b)}$

thus $V = \frac{V_0 \ln(r)}{\ln(a/b)} - \frac{V_0 \ln(b)}{\ln(a/b)}$

$$\Leftrightarrow \boxed{V = \Delta V = V_0 \frac{\ln(r/b)}{\ln(a/b)}}$$

We could differentiate V or integrate E_r to get the relationship between λ_{wire} and ΔV (V_0).

<u>What is the relationship between V_0 and I_{wire}</u> ?

<u>Roadmap</u> :
1. Get $\vec{E}$ in term of V_0.
2. Find relationship between $\vec{E}$ & $\vec{B}$ → get $\vec{B}$ in terms of V_0
3. Compare $\vec{B}_\phi$ and $\vec{B}(V_0)$ → get relationship between V_0 & I !

$\nabla_{x,y}$ or $\nabla_{r,\phi}$ (2D only)

Step#1: $\vec{E}_{xy} = -\nabla V$

$\Leftrightarrow \vec{E}_{xy} = -\left(\frac{\partial V}{\partial r}\hat{r} + \cancel{\frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\phi}}\right)$ 0

$\Rightarrow \vec{E}_{xy} = -\frac{V_0}{\ln(a/b)} \frac{1}{(r/\cancel{b})} \cancel{\frac{1}{b}} \hat{r}$

$$\Rightarrow \boxed{\vec{E}_{xy} = \frac{-V_0}{\ln(a/b)} \frac{1}{r} \hat{r}}$$

Step #2: We found earlier that $\begin{cases} B_x(x,y) = -\frac{n}{c} E_y(x,y) \\ B_y(x,y) = \frac{n}{c} E_x(x,y) \end{cases}$

$$\Leftrightarrow \vec{B}_{xy} = \frac{n}{c} \hat{z} \times \vec{E} = \frac{\vec{k}}{\omega} \times \vec{E}$$

Thus $\vec{B}_{xy} = \frac{n}{c}\left(\frac{-V_0}{\ln(a/b)}\right) \frac{1}{r} \underbrace{\hat{z} \times \hat{r}}_{\hat{\phi}}$

$$\Rightarrow \boxed{\vec{B}_{xy} = \frac{n}{c} \frac{V_0}{\ln(b/a)} \frac{1}{r} \hat{\phi}}$$

Step#3: Also $\vec{B}_\phi = \frac{\mu I_{wire}}{2\pi} \frac{1}{r} \hat{\phi} \equiv \vec{B}_{xy}$

thus $\frac{n}{c} \frac{V_0}{\ln(b/a)} \cancel{\frac{1}{r}} \cancel{\hat{\phi}} = \frac{\mu I_{wire}}{2\pi} \cancel{\frac{1}{r}} \cancel{\hat{\phi}}$

$$\Rightarrow V_0 = \underbrace{\frac{c}{n} \frac{\mu}{2\pi} \ln(b/a)}_{Z_{coax}} I_{wire}$$

$\Rightarrow$ We can associate an impedance with a TEM wave travelling in a coaxial cable: $\boxed{Z_{coax} = \frac{c}{n} \frac{\mu}{2\pi} \ln(b/a)}$ units of Ω.

<u>Example:</u> Standard BNC coaxial cable is RG-58

$$n \simeq 1.5 \qquad \begin{cases} 2b \simeq 3.5\ mm \\ 2a \simeq 1\ mm \end{cases}$$

$$\mu \simeq \mu_0 \simeq 4\pi \times 10^{-7} \quad N/A^2$$

$$\Rightarrow Z_{coax} = \frac{\cancel{3}^{2} \times 10^{\cancel{8}}}{\cancel{1.5}} \frac{\cancel{4\pi}^{2} \times \cancel{10^{-7}}}{\cancel{2\pi}} \underbrace{\ln\left(\frac{3.5}{1}\right)}_{\sim 1.25}$$

$$\simeq 40 \times 1.25$$

$$= 50\,\Omega$$

$\Rightarrow$ a 1A current wave has a corresponding 50 V voltage wave that travels along with it in the cable (in phase).

Impedance Matching Demo

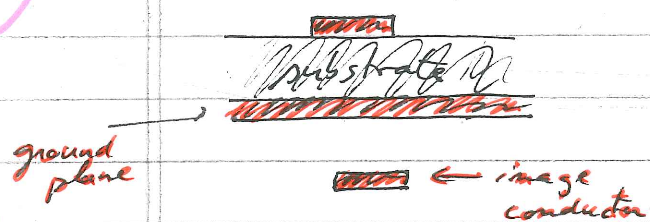
25 ft + 25 ft = 50 ft

L 2 × 50 ft = 100 ft

$$\Rightarrow \begin{cases} \Delta L = 30.5\ m \\ \Delta t = \end{cases}$$

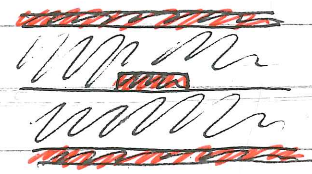
I did NOT do this

note: Any set of parallel conductors can form a transmission line. The only requirement is that the total current adds to zero.



microstrip transmission line

coplanar "waveguide" transmission line



stripline transmission line

I did this

Question: $V_{coax} = Z_{coax} I_{coax} \Rightarrow \text{Power loss} = \dfrac{V_{coax}^2}{Z_{coax}} = I_{coax}^2 Z_{coax}$

But there isn't any resistive heating, so what's happening?

↳ Answer: Energy is travelling down the coax cable away from you $\longrightarrow$ power loss.