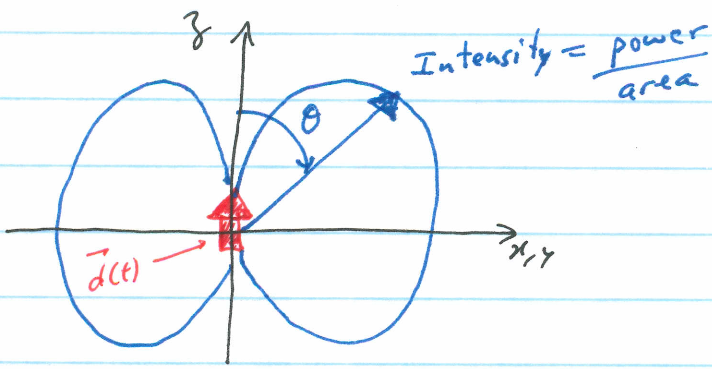
Thursday, February 1, 2024

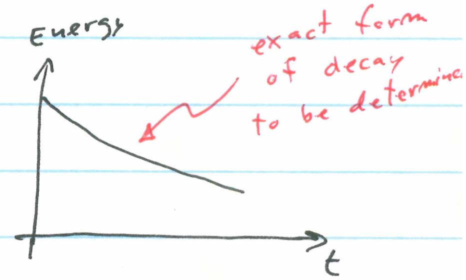
Simple Harmonic Atom – continued

(III) Radiation Damping

Accelerating charges and oscillating dipoles (e.g. antennas) emit radiation at their {natural, driving} frequency ω.

Consider the dipole $\vec{d}(t) = \vec{d}_o \sin(\omega t)$.





$$\boxed{\begin{aligned} \text{Intensity} &= \langle \text{Poynting vector} \rangle = \langle \vec{S} \rangle = \left\langle \frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right)\right\rangle \\ &= \frac{d_o^2}{32\pi\varepsilon_o}\,\frac{\omega^4}{c^3}\,\frac{\sin^2\theta}{r^2}\,\hat{r} \end{aligned}}$$

The total radiated power for an oscillating dipole is

$$\frac{d\langle \text{Energy}\rangle}{dt} = \langle \text{power} \rangle = -\int_0^{\pi}\int_0^{2\pi} \text{Intensity}\, d\phi\, r^2 \sin\theta\, d\theta$$

$$= -\frac{1}{4\pi\varepsilon_o}\,\frac{d_o^2\omega^4}{3c^3} = -\frac{1}{4\pi\varepsilon_o}\,\frac{e^2 r_o^2\,\omega^4}{3c^3}$$

(see Griffiths p. 401-405) here $d_o = e r_o$

what's the form of the damping?

Assume that the damping rate is much smaller than the oscillation rate.

$\omega_0 = \sqrt{\frac{k}{m}}$

In 1D : E_{HO} = Energy of "harmonic" atom $= \frac{1}{2} k r_0^2 = \frac{1}{2} m \omega_0^2 r_0^2$

amplitude of harmonic atom

Energy loss rate $= \frac{d E_{HO}}{dt}$

$$= -\frac{1}{4\pi\varepsilon_0} \frac{e^2 \omega_0^4 r_0^2}{3c^3} = -\frac{1}{4\pi\varepsilon_0} \frac{e^2 \omega^2}{3c^3} \frac{\frac{m}{2}\omega_0^2 r^2}{m/2}$$

E_{HO}

$$= -2 \underbrace{\frac{1}{4\pi\varepsilon_0} \frac{e^2 \omega_0^2}{3mc^3}}_{\gamma} E_{HO}$$

$$\Leftrightarrow \frac{d E_{HO}}{dt} = -2\gamma E_{HO} \Rightarrow \boxed{E_{HO}(t) = E_{HO}\Big|_{t=0} e^{-2\gamma t}}$$

↳ decay is exponential

If $\gamma \ll \omega$, then $r_0(t)$ varies slowly compared to oscillator.

$$\Rightarrow r_0(t) = \sqrt{\frac{2 E_{HO}(t)}{m\omega_0^2}} = \sqrt{\frac{2 E_{HO}(t=0)}{m\omega_0^2} e^{-\gamma t}}$$

$$= r_0 e^{-\gamma t}$$

damping & oscillation are decoupled/independent

$\Rightarrow$ electron oscillates as $\vec{r}(t) = r_0(t) \sin(\omega_0 t) = r_0 e^{-\gamma t} \sin(\omega_0 t)$

The radiation damping result suggests that the model for our semi-classical atom should include radiation damping as a "viscosity" or "friction" force

$$\boxed{\vec{F} = m\ddot{\vec{r}} = -k\vec{r} - 2m\gamma\dot{\vec{r}}}$$

not the only way to include radiation damping in our model.

↳ EM suggests "$\dddot{r}$" term

where $\boxed{\gamma_{classical} = \frac{1}{4\pi\varepsilon_0}\frac{e^2\omega_0^2}{3mc^3}}$

<u>note</u>: QM gives $\gamma_{QM} \propto \omega_0^3$

↳ if you include QM properties of atom (i.e. dipole matrix element) into this classical model, then you get $\gamma \propto \omega^3$

<u>Example</u>: Rubidium atom $\lambda = 780$ nm

$\Rightarrow \omega = 2\pi \times 3.8 \times 10^{14}$ rads/s

We assume that $\omega \gg \gamma$ is valid. Also, we ignored that $\omega \to \sqrt{\omega^2 - \gamma^2}$

↳ $2\gamma_{classical} = 2\pi \times 5.8$ MHz

$$\underbrace{\text{Compare with } 2\gamma_{exp.} = 2\pi \times 6.0 \text{ MHz}}$$

Fortuitous agreement! In <u>H</u> it's a factor 2.4 too big.

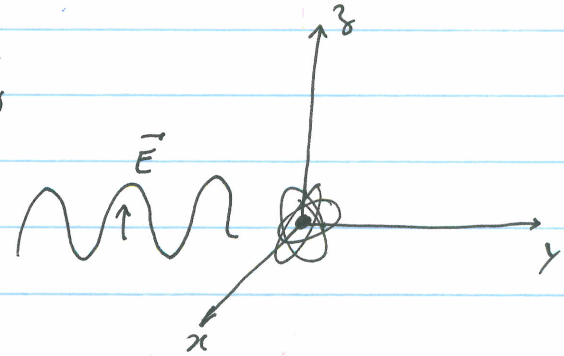
$$\boxed{\text{Demo: You can use (Rayleigh) scattering to determine a powerful, visible laser's polarization.}}$$

Electromagnetically Driven Atom

(i.e. let's shine a laser on our atom)

Consider an EM plane wave with E-field:

$$\vec{E}(t) = E_0 \cos(ky - \omega t)\hat{z}$$



atom is at origin
$(y=0)$

$$\Rightarrow \quad \vec{E}(t) = E_0 \cos(\omega t)\hat{z}$$
$$= \frac{1}{2} E_0 \left[e^{i\omega t} + e^{-i\omega t} \right] \hat{z}$$

If we neglect the magnetic term, then the Lorentz force is given by

$$\vec{F}_{Lorentz} = -e\vec{E}(t)$$

The equation of motion for the atom becomes:

$$\vec{F}_{total} = m\ddot{\vec{r}} = -m\omega_0^2 \vec{r} - 2m\gamma \dot{\vec{r}} - e\vec{E}(t)$$

since motion is along z-axis (we are ignoring $\vec{B}$), then

$$\ddot{z} + 2\gamma \dot{z} + \omega_0^2 z = -\frac{e}{m} E_0 \cos(\omega t)$$

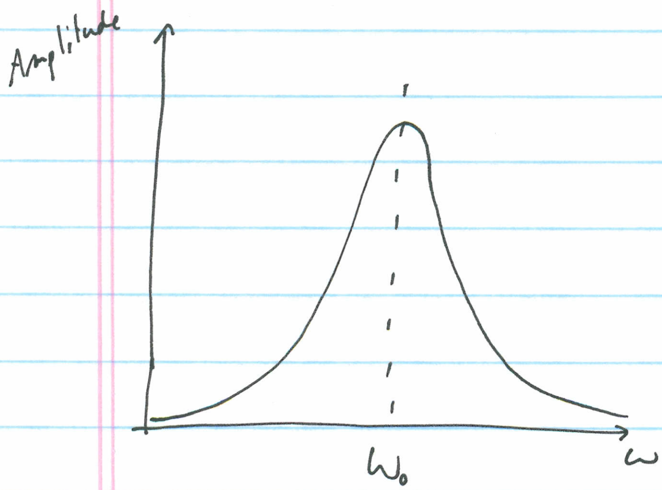
The atom + laser is a driven harmonic oscillator!

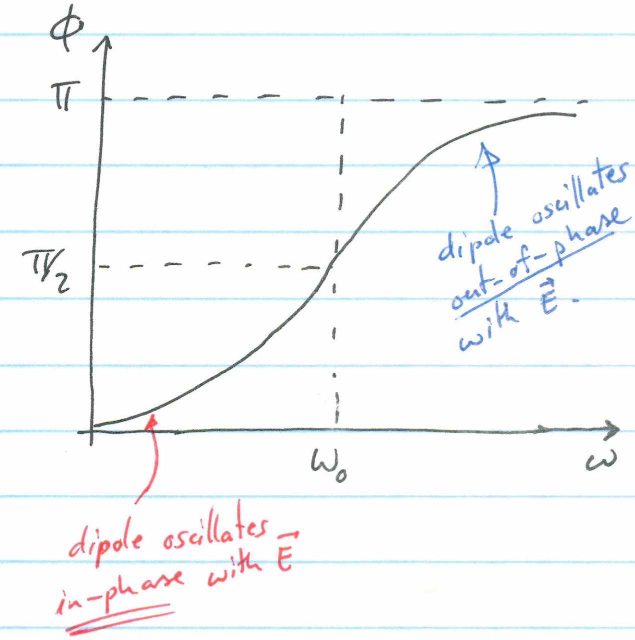
↳ standard class-mech problem!

Ignoring the transient term, the solution is

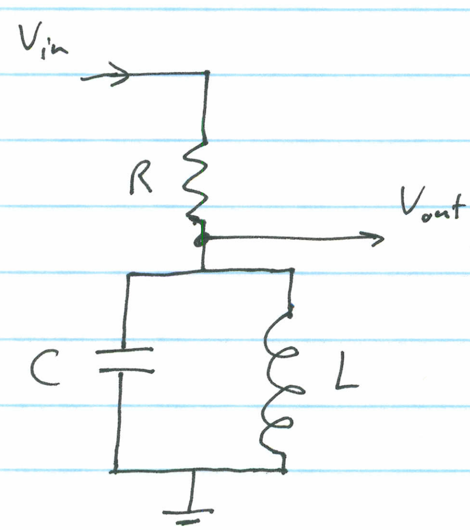
$$\boxed{z(t) = \frac{-e/m \; E_0}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\omega^2\gamma^2}} \cos(\omega t - \phi) \quad \text{with } \phi = \arctan\left(\frac{2\omega\gamma}{\omega_0^2-\omega^2}\right)}$$

see Marion & Thornton chpt 3 (3rd ed.)





Same behavior as an RLC circuit filter



$z(t) \longrightarrow V_{out}(t)$

$E(t) \longrightarrow V_{in}(t)$