Thursday, April 25, 2024

Quantization of the free E-M field

(i.e. the QM description of light, i.e. photons)

I) Classical E-M background

E-M fields in empty space : $\begin{cases} \rho = 0 & \text{no charges} \\ \vec{J} = 0 & \text{no currents} \end{cases}$

Maxwell's Equations :

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 0 & (i) \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & (ii) \\ \vec{\nabla}\cdot\vec{B} = 0 & (iii) \\ \vec{\nabla}\times\vec{B} = \underbrace{\mu_0 \varepsilon_0}_{1/c^2} \dfrac{\partial \vec{E}}{\partial t} & (iv) \end{cases}$$

Lorentz Force law : $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$

E-M potentials :

$$\begin{cases} \vec{E} = -\vec{\nabla} V - \dfrac{\partial \vec{A}}{\partial t} & (a) \\ \vec{B} = \vec{\nabla}\times\vec{A} & (b) \end{cases}$$

substituting (a) into (i), we get

$$-\vec{\nabla}\cdot\vec{\nabla} V - \vec{\nabla}\cdot\frac{\partial \vec{A}}{\partial t} = 0 \iff -\nabla^2 V - \frac{\partial}{\partial t}\Big(\overbrace{\cancel{\vec{\nabla}\cdot\vec{A}}}^{=0 \text{ in Coulomb gauge}}\Big) = 0$$

Thus $\nabla^2 V = 0 \Rightarrow \boxed{V = 0}$ in Coulomb gauge for b.c. $V(r\to\infty) = 0$ (no charges)

Substituting (a) & (b) into (iv), we get

$$\underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{A})}_{=\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{A}}_{=0})-\nabla^2\vec{A}} = -\frac{1}{c^2}\frac{\partial}{\partial t}(\underbrace{\vec{\nabla}V}_{=0}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}$$

in Coulomb gauge

Thus, we get a wave equation for $\vec{A}$:

$$\boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = 0}$$

II Basic Quantization strategy

- the Hamiltonian should be linear in the photon number
  $\hookrightarrow$ $H \sim \hbar\omega N$ $\longrightarrow$ looks like a harmonic oscillator

  ↑ number of photons

reminder: $H_{\text{harmonic oscillator}} = \hbar\omega\left(\hat{N}+\frac{1}{2}\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right) = \frac{\hbar\omega}{2}\left(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger\right)$

- We will use $\vec{A}$ because then there is only one field to worry about.

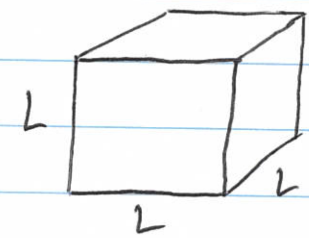
- In the original (basic) quantum treatment of photons (Einstein), the quantities of interest are frequency ω and wavevector $\vec{k}$ of the light, NOT t & x.

  i.e. momentum

  decompose $\vec{A}$ in a plane wave basis

III Fourier expansion of the classical E-M field

Consider the universe as a large box with side L and volume V:



We consider running wave solutions $\longrightarrow$ the classically permitted [periodic boundary conditions: $\vec{A}(x=0) = \vec{A}(x=L)$] modes are quantized.

[not quantum quantized!]

The solutions for $\vec{A}$ are

$$\vec{A}_{k,s} = A_{k,s}\, e^{i(\vec{k}\cdot\vec{r} - \omega_k t)}\, \hat{\varepsilon}_{k,s} + \underbrace{c.c.}_{A^*_{k,s}\, e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\, \hat{\varepsilon}_{k,s}}$$

note: $\hat{\varepsilon}_{k,s}$ define the $\underbrace{\text{polarization direction}}_{\text{(linear or circular basis)}}$ with: $\begin{cases} \hat{\varepsilon}_{k,s}\cdot\vec{k} = 0 \\ \hat{\varepsilon}_{k,s=1}\cdot\hat{\varepsilon}_{k,s=2} = 0 \end{cases}$

Here, $\omega_k = ck$ and $k_{x,y,z} = \dfrac{2\pi\, n_{x,y,z}}{L}$ with $n_{x,y,z} = 0, \pm 1, \pm 2, \ldots$

Thus, the general solution is

$$\vec{A}(\vec{r},t) = \sum_{k,\, s=1,2} \vec{A}_{k,s}$$

$$= \sum_{k,\, s=1,2} \hat{\varepsilon}_{k,s} \left\{ A_{k,s}\, e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + A^*_{k,s}\, e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)} \right\}$$

Amplitude coefficients for planewave states thus

$$\begin{cases} \vec{E} = -\dfrac{\partial A}{\partial t} = \displaystyle\sum_{k,\, s=1,2} \hat{\varepsilon}_{k,s} \left\{ i\omega_k A_{k,s} e^{i(k\cdot\vec{r} - \omega_k t)} - i\omega_k A^*_{k,s} e^{-i(k\cdot\vec{r} - \omega_k t)} \right\} \\ \vec{B} = \vec{\nabla}\times\vec{A} = \displaystyle\sum_{k,\, s=1,2} ik \times \hat{\varepsilon}_{k,s} \left\{ A_{k,s} e^{i(k\cdot\vec{r} - \omega_k t)} - A^*_{k,s} e^{-i(k\cdot\vec{r} - \omega_k t)} \right\} \end{cases}$$

IV Quantization by Analogy

The energy and thus Hamiltonian is given by

$$H \equiv E = \frac{1}{2}\int dV \left\{ \varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2 \right\} = \sum_{k,s} \left(p^2_{k,s} + \omega_k^2\, q^2_{k,s} \right)$$

abstract {momentum, position}

really, $H = \sum_i p_i \dot{q}_i - \mathcal{L}$ with $p_i = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}$ and $[p_i, q_i] = -i\hbar$

and $\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\left(E^2 - \frac{B^2}{c^2}\right)$ with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

[Lorentz Gauge]

$\rightarrow$ but, we're taking a shortcut

note: $$\int_V dV \cdot e^{\pm i(k - k')\cdot\vec{r}} = V\,\delta_{k,k'}$$

After some math (substitute for $\vec{E}$ & $\vec{B}$) and lots of Algebra, we get ...

$$H_{classical} = \varepsilon V \sum_{k,s} \omega_k^2 \left\{ A_{k,s} A^*_{k,s} + A^*_{k,s} A_{k,s} \right\}$$

for comparison $H_{\substack{\text{quantum}\\ \text{harmonic}\\ \text{oscillator}}} = \frac{1}{2}\hbar\omega\left(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\right)$

for many independent harmonic oscillators:

$$H_{quantum} = \sum_i \frac{1}{2}\hbar\omega\left(\hat{a}_i\hat{a}_i^+ + \hat{a}_i^+\hat{a}_i\right)$$

↳ we postulate that the Hamiltonian for the free EM field is

$$\boxed{\begin{gathered} H_{\substack{\text{photon}\\ \text{field}}} \equiv \sum_{k,s} \frac{1}{2}\hbar\omega_{k,s} \left\{ \hat{a}_{k,s}\hat{a}^+_{k,s} + \hat{a}^+_{k,s}\hat{a}_{k,s} \right\} \\ \text{with } A_{k,s} \equiv \sqrt{\frac{\hbar}{2\varepsilon\omega_k V}}\,\hat{a}_{k,s}\,, \quad A^*_{k,s} \equiv \sqrt{\frac{\hbar}{2\varepsilon\omega_k V}} \end{gathered}}$$

Quantum states of light

A- Fock states basis

the eigenstates of the harmonic oscillator are the photon occupation states

$|\psi\rangle = |n_{k,s}\rangle$ = n photons in the mode with k and polarization s.

"no photons" is still a state $|0_{\vec{k},s}\rangle$ with energy $\frac{\hbar\omega_k}{2}$

↳ called the vacuum state.

more generally multiple states can be simultaneously occupied:

$$|\psi\rangle = |\{n\}\rangle = \underbrace{|n_{k_1,\uparrow}\rangle |n_{k_1,\rightarrow}\rangle \cdots |n_{k_m,\uparrow}\rangle |n_{k_m,\rightarrow}\rangle}_{\text{order does not matter}}$$

most general

$$|\psi\rangle = \sum_{k_1 s_1, \ldots, k_m, s_m} C_{k_1 s_1 \ldots k_m, s_m} |n_{k_1,s_1}\rangle \ldots |n_{k_m,s_m}\rangle$$

review:

$$\hat{a}_{\vec{k},s} |n_{\vec{k},s}\rangle = \sqrt{n_{\vec{k},s}}\, |n_{\vec{k},s} - 1\rangle$$

$$\hat{a}^{+}_{\vec{k},s} |n_{\vec{k},s}\rangle = (n_{\vec{k},s} + 1)\, |n_{\vec{k},s} + 1\rangle$$

$$\hat{N}_{\vec{k},s} |n_{\vec{k},s}\rangle = \hat{a}^{+}_{\vec{k},s}\, \hat{a}_{\vec{k},s} |n_{\vec{k},s}\rangle = n_{\vec{k},s} |n_{\vec{k},s}\rangle$$

commutation relations:

$$[\hat{a}_{\vec{k},s}, \hat{a}^{+}_{\vec{k}',s'}] = \delta_{\vec{k},\vec{k}'}\, \delta_{s,s'}$$

$$[\hat{a}_{\vec{k},s}, \hat{a}_{\vec{k},s}] = 0$$

$$[\hat{a}^{+}_{\vec{k},s}, \hat{a}^{+}_{\vec{k},s}] = 0$$