Tuesday, April 30, 2024

## Fock states (continued)

Fock state: $|\{n\}\rangle = \underbrace{|n_{k_1,\uparrow}\rangle |n_{k_1,\rightarrow}\rangle \cdots |n_{k_m,\uparrow}\rangle |n_{k_m,\rightarrow}\rangle}_{\text{order does } \underline{\text{not}} \text{ matter}}$

creation & annihilation operators:

$$\hat{a}_{k,s} |n_{k,s}\rangle = \sqrt{n_{k,s}}\, |n_{k,s} - 1\rangle$$

$$\hat{a}^{+}_{k,s} |n_{k,s}\rangle = \sqrt{n_{k,s}+1}\, |n_{k,s} + 1\rangle$$

We can write any Fock state as

$$|\{n\}\rangle = \prod_{k,s} \frac{\left(\hat{a}^{+}_{k,s}\right)^{n_{k,s}}}{\sqrt{n_{k,s}!}} |0\rangle$$

The Fock states form a complete basis for the electromagnetic field

A photon is an excitation of the quantum electromagnetic $\underbrace{\text{"potential"}}_{\text{Hamiltonian}}$

## What's the electric field of a Fock state photon?

Heisenberg representation

$${}_{k,s}\langle n|\vec{E}|n\rangle_{k,s} =$$

$${}_{k,s}\langle n \sum_{k',s'} \varepsilon_{k',s'} \left\{ i\omega_{k'} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_{k'} V}}\, \hat{a}_{k',s'}\, e^{i(k'\cdot \vec{r} - \omega_{k'} t)} - i\omega_{k'} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_{k'} V}}\, \hat{a}^{+}_{k',s'}\, e^{-i(k'\cdot\vec{r} - \omega_{k'} t)} \right\} |n\rangle_{k,s}$$

$${}_{k,s}\langle n|\, \hat{a}_{k',s'}\, |n\rangle_{k,s}$$

$$= {}_{k,s}\langle n|\, \delta_{k,k'}\, \delta_{s,s'} \sqrt{n_{k,s}}\, |n_{k,s} - 1\rangle$$

$$= 0$$

$${}_{k,s}\langle n|\, \hat{a}^{+}_{k',s'}\, |n\rangle_{k,s}$$

$$= {}_{k,s}\langle n|\, \delta_{k,k'}\, \delta_{s,s'} \sqrt{n_{k,s}+1}\, |n+1\rangle_{k,s}$$

$$= 0$$

so $\boxed{{}_{k,s}\langle n|\vec{E}|n\rangle_{k,s} = 0}$ (not time averaged!!)

What's the variance $\Delta E$ of the electric field?

$$\Delta E^2 = \langle n|E^2|n\rangle - (\underbrace{\langle n|E|n\rangle}_{=0})^2 \quad \simeq \text{Intensity in this case}$$

$$= {}_{k,s}\langle n| \left\{ \sum_{k',s'} (cst\ \hat{a}_{k',s'} + cst^{*}\ \hat{a}^{+}_{k',s'}) \right\}^2 |n\rangle_{k,s}$$

the only possible non-zero terms are

$$\langle n|\, |cst|^2 \underbrace{\hat{a}_{k,s}\hat{a}^{+}_{k,s}}_{\sqrt{n+1}\sqrt{n+1}} + |cst|^2 \underbrace{\hat{a}^{+}_{k,s}\hat{a}_{k,s}}_{\sqrt{n}\sqrt{n}} + (cst)^2 \underbrace{\hat{a}_{k,s}\hat{a}_{k,s}}_{\langle n|\sqrt{n-1}\sqrt{n}|n-2\rangle = 0} + (cst^*)^2 \underbrace{\hat{a}^{+}_{k,s}\hat{a}^{+}_{k,s}}_{\langle n|\sqrt{n+2}\sqrt{n+1}|n+ \;= 0} |n\rangle$$

$$= \langle n|\, |cst|^2(n+1) + |cst|^2(n)\, |n\rangle$$

$$= |cst|^2(2n+1)$$

Thus,

$$\boxed{\langle n|\Delta E^2|n\rangle = \omega_k^2\left(\frac{\hbar}{2\varepsilon_0 \omega_k V}\right)(2n+1) = \frac{\hbar\omega_k}{2\varepsilon_0 V}(2n+1)}$$

even for $n=0$, there are fluctuations (i.e. vacuum fluctuations)

$\Rightarrow$ Variance (and intensity) is non-zero

$\hookrightarrow$ this approach indicates that $|\vec{E}| = \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\sqrt{2n+1}$

Paradox?

How can the electric field be zero, but the electric field noise (or intensity) be non-zero? (throw out Quantum Field Theory?)
$\hookrightarrow$ No!

Answer: A Fock state is a quantum state of light with no classical analog.

The $|n\rangle$ state produces no time development of $\vec{E}$!
↳ i.e. there is no phase associated with $\vec{E}$!

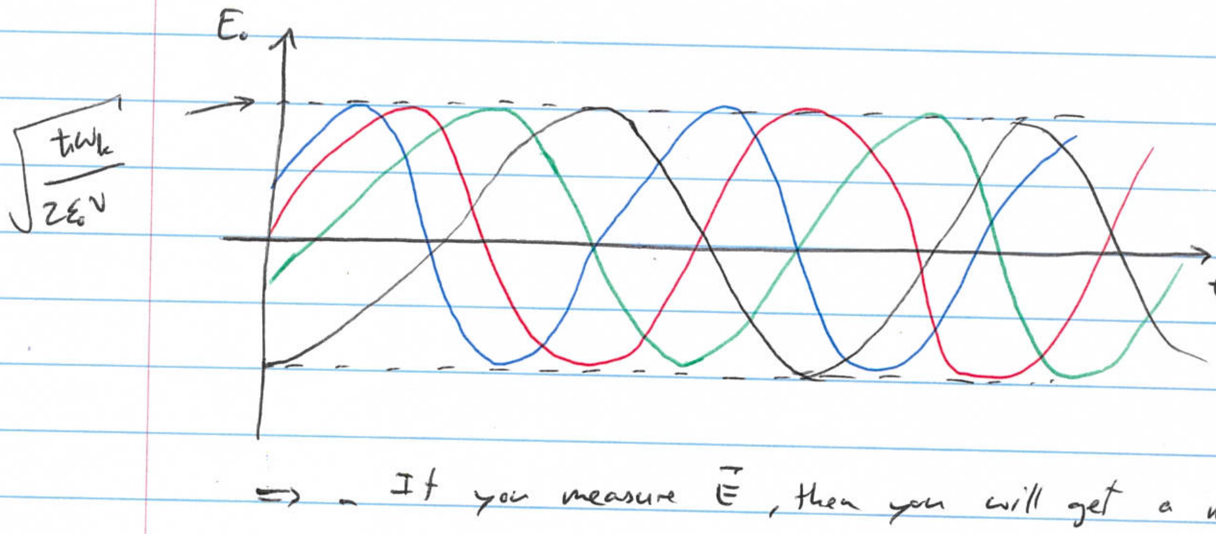
However, photon number $\underline{\underline{n}}$ is completely well defined.

$$\langle n|\hat{N}|n\rangle = n$$

$$\langle n|\Delta N^2|n\rangle = \langle n|\hat{N}^2|n\rangle - \langle n|\hat{N}|n\rangle^2 = n^2 - n^2 = 0$$

In practice, it is difficult to make a Fock state $|n\rangle$ in the lab, except for $|0\rangle$.

<u>physical picture</u>: Electric field is a superposition of many phase states




⇒ - If you measure $\vec{E}$, then you will get a non-zero result.
- $\langle n|\vec{E}|n\rangle = 0$, i.e. multiple measurements average to zero.
- $\langle n|\Delta\vec{E}^2|n\rangle \neq 0$.
- Amplitude is well defined.
- Phase is completely undefined.