Thursday, May 2, 2024

Coherent States

Classical states of light (e.g. laser light) are associated with coherent states:

Definition: Coherent state $|\alpha\rangle$ (α = complex number)

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$ ← normalized $\langle\alpha|\alpha\rangle = 1$

A coherent state is an eigenstate of the annihilation operator

non-Hermitian

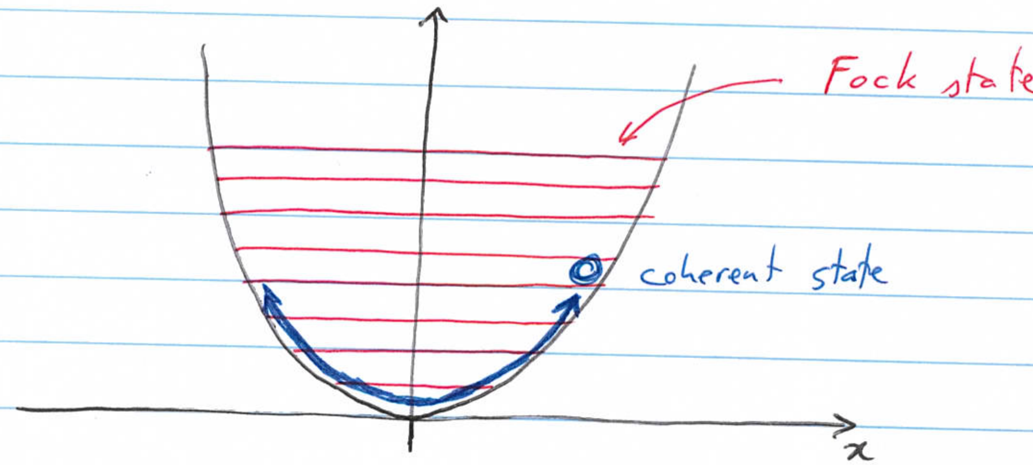
$$\boxed{\begin{aligned} \hat{a}|\alpha\rangle &= \alpha|\alpha\rangle \\ \langle\alpha|\hat{a}^{+} &= \langle\alpha|\alpha^{*} \end{aligned}}$$

Coherent states are not generally orthogonal to each other.

↳ do NOT form a basis.

$\langle\alpha|\beta\rangle \neq 0$, actually $|\langle\alpha|\beta\rangle|^2 = \exp(-|\alpha-\beta|^2)$

A coherent state of a harmonic oscillator represents the probability amplitude (i.e. wavefunction) for a classical particle oscillating in a harmonic potential.



<u>Note:</u> A coherent state also obeys the minimum Heisenberg uncertainty relation $\Delta x \Delta p = \frac{\hbar}{2}$

<u>properties:</u> $\langle n \rangle = {}_{k,s}\langle \alpha | \hat{N}_{k,s} | \alpha \rangle_{k,s}$

$$= {}_{k,s}\langle \alpha | \hat{a}^{+}_{k,s} \hat{a}_{k,s} | \alpha \rangle_{k,s} = |\alpha|^2$$

Also, $\Delta N^2 = {}_{k,s}\langle \alpha | \underbrace{\hat{N}^2_{k,s}} | \alpha \rangle_{k,s} - \left({}_{k,s}\langle \alpha | \hat{N}_{k,s} | \alpha \rangle_{k,s} \right)^2$

$$a^+ a a^+ a = a^+ (a^+ a + 1) a$$
$$= a^+ a^+ a a + a^+ a$$
$$= :a^+ a a^+ a:$$

$:\hat{O}: =$ normal ordering

$$= \alpha^* \alpha^* \alpha \alpha + \alpha^* \alpha - \langle n \rangle^2$$
$$= |\alpha|^4 + |\alpha|^2 - \langle n \rangle^2$$
$$= \langle n \rangle^2 + \langle n \rangle - \langle n \rangle^2$$
$$= \langle n \rangle$$

Thus $\langle(\Delta N)^2\rangle = \langle n\rangle \Rightarrow \langle \Delta N\rangle = \sqrt{\langle n\rangle} = |\alpha|$

Coherent states have $\langle n\rangle = |\alpha|^2$ photons on average, and they have a spread of $\pm\sqrt{\langle n\rangle} = \pm|\alpha|$ photons. [coherent states obey poisson statistics.]

↳ The photon number can fluctuate! It is not fixed as in the Fock state.

<u>What's the E-field of a coherent state $|\alpha\rangle$?</u>

$\langle\alpha|\vec{E}|\alpha\rangle$

note: $\alpha = |\alpha|e^{i\phi} \in \mathbb{C}$

$$\langle\alpha|_{\vec{k},s} \sum_{\vec{k}',s'} \left[\hat{\varepsilon}_{\vec{k}',s'}\, i\omega_{k'} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_{k'} V}} \left\{ \underbrace{\hat{a}_{\vec{k}',s'}}_{\alpha = |\alpha|e^{i\phi}} e^{i(\vec{k}'\cdot\vec{r} - \omega_{k'}t)} - \underbrace{\hat{a}^{+}_{\vec{k}',s'}}_{\alpha^* = |\alpha|e^{-i\phi}} e^{-i(\vec{k}'\cdot\vec{r}-\omega_{k'}t)} \right\}\right] |\alpha\rangle_{\vec{k},s}$$

$$= \hat{\varepsilon}_{\vec{k},s}\, |\alpha|\, \cancel{i}^{-1} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \left\{ \frac{e^{i(\vec{k}\cdot\vec{r}-\omega_k t+\phi)} - e^{-i(\vec{k}\cdot\vec{r}-\omega_k t-\phi)}}{2i} \right\} 2$$

$$= -\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\, 2|\alpha| \sin(\vec{k}\cdot\vec{r} - \omega_k t + \phi)\cdot \hat{\varepsilon}_{\vec{k},s}$$

↳ classical oscillating electric field!!!

coherent state describes laser field, oscillating dipole radiation

The amplitude can also include contributions from several wavevectors in order to get a more spatially localized wavepacket.

What's the variance $(\Delta E)^2$ of $\vec{E}$?

$$\langle \Delta E^2 \rangle = \langle \alpha | \vec{E}^2 | \alpha \rangle - \left(\langle \alpha | \vec{E} | \alpha \rangle \right)^2$$

$$= \langle \alpha | \vec{E}^2 | \alpha \rangle - 4|\alpha|^2 \frac{\hbar \omega_k}{2\varepsilon V} \sin^2(\vec{k}\cdot\vec{r} - \omega_k t + \phi)$$

we must calculate $\langle \alpha | \vec{E}^2 | \alpha \rangle$

$$= {}_{\vec{k},s}\langle \alpha | \left[\sum_{\vec{k}',s'} \hat{\varepsilon}_{\vec{k}',s'}\, i\omega_{k'} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_{k'} V}} \left\{ \hat{a}_{\vec{k}',s'} e^{i(\vec{k}'\cdot\vec{r} - \omega_{k'} t)} - \hat{a}^{\dagger}_{\vec{k}',s'} e^{-i(\vec{k}'\cdot\vec{r} - \omega_{k'} t)} \right\} \right]^2 | \alpha \rangle_{\vec{k},s}$$

only $\left\{ \begin{matrix} \vec{k}' = \vec{k} \\ s' = s \end{matrix} \right\}$ terms contribute $\rightarrow$ get rid of sum

$$= \cancelto{1}{\hat{\varepsilon}^2_{\vec{k},s}} \frac{-\hbar \omega_k}{2\varepsilon_0 \omega_k V} \langle \alpha | \hat{a}\hat{a} e^{2i(\vec{k}\cdot\vec{r} - \omega_k t)} + \hat{a}^{\dagger}\hat{a}^{\dagger} e^{-2i(\vec{k}\cdot\vec{r} - \omega_k t)} - \underbrace{\hat{a}\hat{a}^{\dagger}}_{1 + a^{\dagger}a} - \hat{a}^{\dagger}\hat{a} | \alpha \rangle$$

$$= \frac{-\hbar \omega_k}{2\varepsilon \omega_k V} \left\{ \underbrace{|\alpha|^2 2 \left(\frac{e^{2i(\vec{k}\cdot\vec{r} - \omega_k t + \phi)} + e^{-2i(\vec{k}\cdot\vec{r} - \omega_k t + \phi)}}{2} \right)}_{2|\alpha|^2 \cos\left[2(\vec{k}\cdot\vec{r} - \omega_k t + \phi)\right]} - 1 - 2|\alpha|^2 \right\}$$

⋮ a little bit of algebra

$$= \frac{\hbar \omega_k}{2\varepsilon V} \left[4|\alpha|^2 \sin^2(\vec{k}\cdot\vec{r} - \omega_k t + \phi) + 1 \right]$$

Thus, $$\langle \Delta E^2 \rangle = \cancel{\frac{\hbar \omega_k}{2\varepsilon_0 V} 4|\alpha|^2 \sin^2(\vec{k}\cdot\vec{r} - \omega_k t + \phi)} + \frac{\hbar \omega_k}{2\varepsilon_0 V} \cancel{- 4|\alpha|^2 \frac{\hbar \omega_k}{2\varepsilon_0 V} \sin^2(\vec{k}\cdot\vec{r} - \omega_k t + \phi)}$$

$$\Rightarrow \quad \langle \Delta E^2 \rangle = \frac{\hbar \omega}{2\varepsilon_0 V} \quad \Rightarrow \left\{ \begin{matrix} \text{independent of } |\alpha| \text{ and } \phi\ ! \\ \text{no time dependence!} \end{matrix} \right.$$

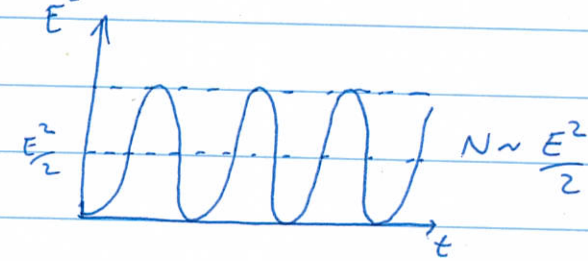
so $\Delta E = \sqrt{\dfrac{\hbar\omega}{2\varepsilon_0 V}}$

<u>note</u>: This is the same electric field noise as for the $|0\rangle$ vacuum state (note: $|\alpha=0\rangle = |0\rangle$ Fock stat

This result seems to contradict the calculation of $\underbrace{\langle N \rangle}_{\langle n\rangle = |\alpha|^2}$ and $\underbrace{\Delta N}_{\sqrt{\langle n \rangle} = |\alpha|}$, but it doesn't:

electric field: $|\vec{E}| \sim |\alpha| \sim \sqrt{n}$

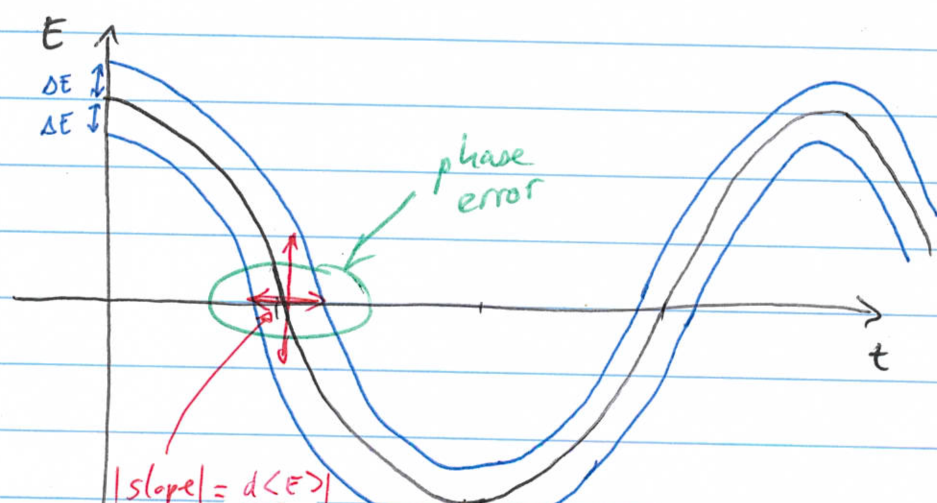
$\Delta E \sim 1$



Error on $|\vec{E}|^2 \sim$ error on intensity $I \sim$ error on photon number N

$$\Delta N = \frac{1}{2}\left[(E+\Delta E)^2 - (E-\Delta E)^2\right]$$
$$= \frac{1}{2} \quad 2 \times 2 E \Delta E \quad = \sqrt{n}$$

$\Rightarrow$ Uncertainty in N appears to come from vacuum ($|0\rangle$) $\vec{E}$-field fluctuations.



$|\text{slope}| = \left.\dfrac{d\langle E\rangle}{dt}\right|_{t=T/4}$

see page #3

$= \sqrt{\dfrac{\hbar\omega}{2\varepsilon_0 V}}\, 2|\alpha|\omega$

<u>what is the phase error</u>

$$\Delta t = \frac{\Delta E}{\text{slope}} = \frac{\cancel{\sqrt{\hbar\omega/2\varepsilon_0 V}}}{\cancel{\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}}\, 2|\alpha|\omega}$$

$$\Rightarrow \Delta t = \frac{1}{\omega\, 2|\alpha|}$$

$$\Rightarrow \Delta\phi = \omega \Delta t = \frac{1}{2|\alpha|}$$

Notably, we have $\Delta n \Delta\phi = |\alpha| \dfrac{1}{2|\alpha|} = \dfrac{1}{2}$

A universal phase operator is hard to define for all quantum states, but in general we have

$$\boxed{\Delta n \, \Delta\phi \geq \frac{1}{2}}$$ for any photon state

Recall: For a Fock state, we have $\Delta n = 0$, but $\Delta\phi \rightarrow 2\pi$
(or $\pm\infty$)

Note: The analysis suggests that the uncertainty in the E-field (and hence photon number) is due to the quantum electromagnetic vacuum $|0\rangle$.