Thursday, February 22, 2024

Reminder: Hamiltonian for a 2-level atom interacting with an EM field

$$H = \underbrace{\begin{array}{c|cc} & |g\rangle & |e\rangle \\ \hline \langle g| & E_g & 0 \\ \langle e| & 0 & E_e \end{array}}_{H_0 = \hbar \begin{pmatrix} \omega_g & 0 \\ 0 & \omega_e \end{pmatrix}} + \underbrace{\hbar \begin{array}{c} \\ \end{array}\!\!\begin{array}{cc} |g\rangle & |e\rangle \end{array} \!\!\begin{pmatrix} 0 & \Omega \\ \Omega^* & 0 \end{pmatrix}}_{H_{int}} \cos(\omega_\ell t)$$

Schrodinger Equation (i.e what's the time evolution of the system)

The schrodinger equation is

$$(1) \quad i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle$$

We will find a solution in the unperturbed basis $\{|g\rangle, |e\rangle\}$ of the form:

$$(2) \quad |\psi(t)\rangle = \underbrace{C_g(t)}_{\text{non-trivial time-dependence}} e^{-i\omega_g t} |g\rangle + \underbrace{C_e(t)}_{\text{non-trivial time-dependence}} e^{-i\omega_e t} |e\rangle$$

$\omega_g = E_g/\hbar$, $\omega_e = E_e/\hbar$

(2) $\longrightarrow$ (1) "plug & chug"

$$i\hbar \left[\frac{\partial}{\partial t} C_g(t) \cdot e^{-i\omega_g t} + \cancel{C_g(t) \cdot (-i\omega_g) e^{-i\omega_g t}} \right] |g\rangle$$

$$= \left[\cancel{C_g(t) \cdot e^{-i\omega_g t} E_g} + C_e(t) e^{-i\omega_e t} \hbar\Omega \cos(\omega_\ell t) \right] |g\rangle$$

Thus

$e^{-i\omega_g t}$

$$(3a) \quad i\hbar \frac{d}{dt} C_g(t) = C_e(t)\, \hbar \Omega \cos(\omega_\ell t)\, e^{-i\omega_{eg} t}$$

$\omega_e - \omega_g$

Similarly

$$(3b) \quad i\hbar \frac{d}{dt} C_e(t) = C_g(t)\, \hbar \Omega^* \cos(\omega_\ell t)\, e^{+i\omega_{eg} t}$$

(no approximations so far!)

Rotating Wave Approximation (RWA)

$$(3a) \longrightarrow \cancel{i\hbar} \frac{d}{dt} C_g(t) = C_e(t)\, \cancel{\hbar} \Omega \left[\frac{e^{i\omega_\ell t} + e^{-i\omega_\ell t}}{2} \right] e^{-i\omega_{eg} t}$$

$$\Longleftrightarrow \quad i \frac{d}{dt} C_g(t) = \frac{C_e(t)\, \Omega}{2} \left[\underbrace{e^{i(\omega_\ell - \omega_{eg})t}}_{\text{slow oscillations}} + \underbrace{e^{-i(\omega_\ell + \omega_{eg})t}}_{\text{very fast oscillations}} \right]$$

very fast oscillations → ignore (RWA)

(averages to "zero")

Partial justification: if $C_e(t) \simeq cst$, then integrate $\dot{C}_g(t)$ with respect to t

$$\Rightarrow \quad i\, C_g(t) \simeq \frac{C_e \Omega}{2} \left[\frac{e^{+i(\omega_\ell - \omega_{eg})t}}{i(\omega_\ell - \omega_{eg})} + \frac{e^{-i(\omega_\ell - \omega_{eg})t}}{\underbrace{-i(\omega_\ell + \omega_{eg})}_{\text{very small}}} \right]$$

$$\Rightarrow \begin{cases} i\dfrac{d}{dt}C_g(t) \simeq \dfrac{C_e(t)\Omega}{2} e^{+i\overbrace{(\omega_\ell - \omega_{eg})}^{\delta = \omega_\ell - \omega_{eg} \;=\; \text{detuning}}t} & (4a) \\ \text{similarly,} \\ i\dfrac{d}{dt}C_e(t) \simeq \dfrac{C_g(t)\Omega^*}{2} e^{-i\underbrace{(\omega_\ell - \omega_{eg})}_{\delta}t} & (4b) \end{cases}$$

<u>Rabi Oscillations</u>

we need to get separate equations for $C_g(t)$ & $C_e(t)$

$$\frac{d}{dt}(4a) \Rightarrow i\frac{d^2}{dt^2}C_g(t) = \frac{C_e(t)\Omega}{2}(+i\delta)e^{i\delta t} + \frac{\Omega}{2}e^{i\delta t}\underbrace{\left(\frac{d}{dt}C_e(t)\right)}$$

substitute from (4a) → "$\frac{d}{dt}C_g(t)$" term

substitute from (4b) "$C_g(t)$" term

repeat for $\frac{d}{dt}(4b)$

After differentiation & substitution, we get (see Metcalf & Van der Straten)

$$\begin{cases} \dfrac{d^2}{dt^2}C_g - i\delta\dfrac{d}{dt}C_g + \dfrac{|\Omega|^2}{4}C_g = 0 & (5a) \\ \dfrac{d^2}{dt^2}C_e + i\delta\dfrac{d}{dt}C_e + \dfrac{|\Omega|^2}{4}C_e = 0 & (5b) \end{cases}$$

Easy/straightforward to solve! (but tedious)

(standard 2nd order diff. eq. with constant coefficients)

Initial conditions: atom in ground state

$$c_g(t=0) = 1 \quad , \quad \frac{d}{dt} c_g(t=0) = 0$$

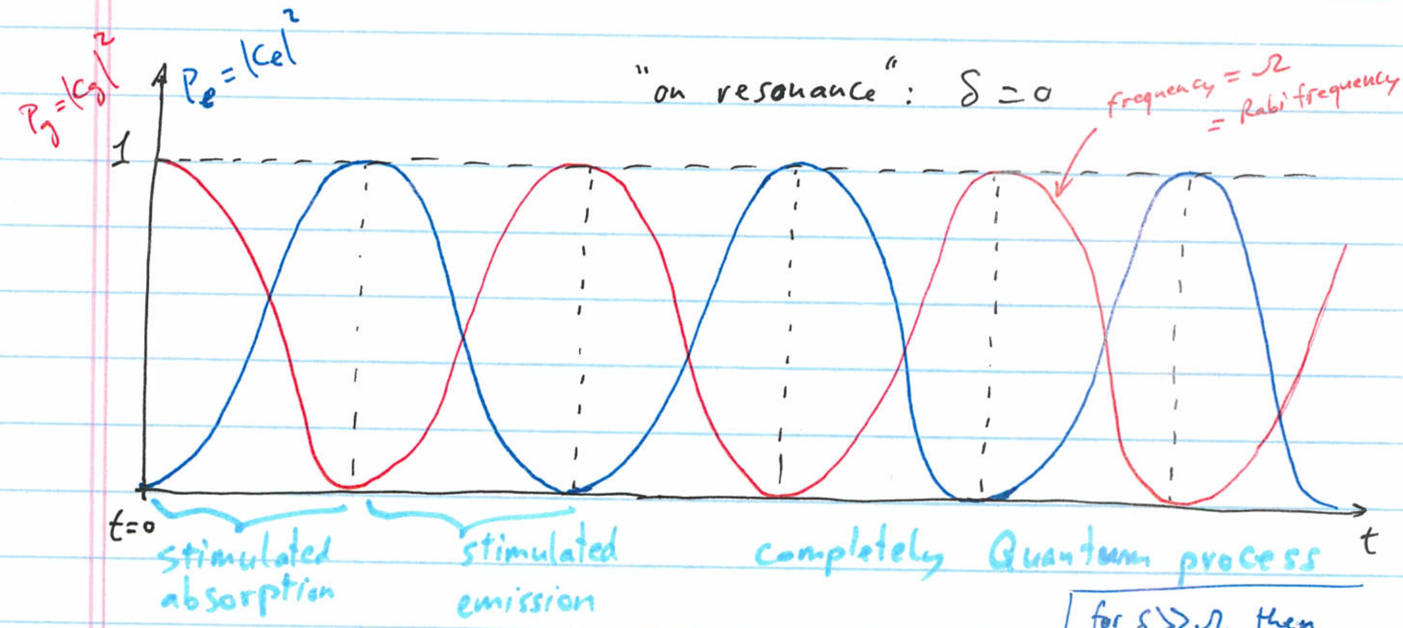
$$c_e(t=0) = 0 \quad , \quad \frac{d}{dt} c_e(t=0) = 0$$

Solution

$$\begin{cases} c_g(t) = \left[\cos\left(\frac{\Omega'}{2}t\right) - \frac{i\delta}{\Omega'} \sin\left(\frac{\Omega' t}{2}\right)\right] e^{+i\frac{\delta t}{2}} & (6a) \\ c_e(t) = -\frac{i\Omega}{\Omega'} \sin\left(\frac{\Omega' t}{2}\right) e^{-i\frac{\delta t}{2}} & (6b) \end{cases}$$

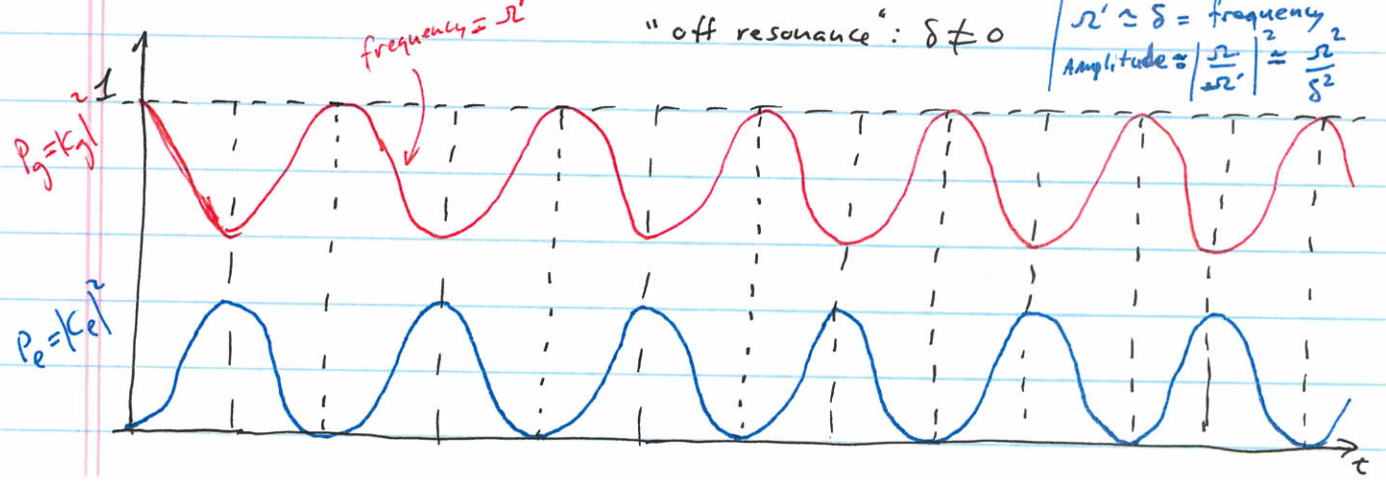
with $\delta = \omega_\ell - \omega_{eg}$ = detuning and $\Omega' = \sqrt{\Omega^2 + \delta^2}$ = generalized Rabi frequency

"on resonance": $\delta = 0$



for $\delta \gg \Omega$ then
$\Omega' \simeq \delta$ = frequency
Amplitude $\simeq \left|\frac{\Omega}{\Omega'}\right|^2 \simeq \frac{\Omega^2}{\delta^2}$

"off resonance": $\delta \neq 0$



Energy of a 2-level atom in an EM field

The Hamiltonian for the system is time-dependent

↳ energy is not conserved, though it is on average.

Let's take a closer look at $C_g(t)$ for $|\delta| \gg |\Omega|$

↳ in this case $\underbrace{|C_g|^2}_{1-\frac{\Omega^2}{\delta^2}} \gg \underbrace{|C_e|^2}_{\frac{\Omega^2}{\delta^2}}$ and $\Omega' \simeq |\delta|$

Thus $|\psi(t)\rangle \simeq C_g(t)\, e^{-i\omega_g t} |g\rangle + \varepsilon |e\rangle$

$$C_g(t) = \left[\cos\left(\frac{\Omega'}{2}t\right) - i\underbrace{\frac{\delta}{\Omega'}}_{\simeq \pm 1 \text{ (sign of } \delta)} \sin\left(\frac{\Omega'}{2}t\right)\right] e^{+i\frac{\delta}{2}t}$$

sign of δ

$$\simeq e^{-i\frac{\Omega'}{2}t}\, e^{+i\frac{\delta}{2}t}$$

$$\simeq e^{-i\left(\cancel{\frac{\delta}{2}} + \frac{1}{4}\frac{\Omega^2}{\delta} - \cancel{\frac{\delta}{2}}\right)t}$$

$$\simeq e^{-i\frac{1}{4}\frac{\Omega^2}{\delta}t}$$

$$\Omega' = \sqrt{\delta^2 + \Omega^2}$$
$$= |\delta|\sqrt{1 + \frac{\Omega^2}{\delta^2}}$$
$$\simeq |\delta|\left(1 + \frac{1}{2}\frac{\Omega^2}{\delta^2} + \cdots\right)$$
$$\simeq |\delta| + \frac{1}{2}\frac{\Omega^2}{|\delta|} + \cdots$$

Thus $|\psi(t)\rangle = e^{-i\left(\omega_g + \frac{1}{4}\frac{\Omega^2}{\delta}\right)t} |g\rangle + \varepsilon|e\rangle$

↳ We can interpret this shift in "oscillation frequency" as meaning ~~that~~ the energy of the ground state is shifted.

Energy shift of ground state: $\boxed{E_g' = E_g + \frac{\hbar\Omega^2}{4\delta}}$ for $|\delta| \gg |\Omega|$