Thursday, February 29, 2024

#1

Dressed atom Hamiltonian (includes off resonance case)

$$H = H_{atom} + H_{photons} + H_{interaction}$$

$$= \begin{pmatrix} E_g & 0 \\ 0 & E_e \end{pmatrix} + \hbar\omega_\ell \begin{pmatrix} N+1 & 0 \\ 0 & N \end{pmatrix} + \hbar \begin{pmatrix} 0 & \Omega/2 \\ \Omega^*/2 & 0 \end{pmatrix}$$

in the $\{|1\rangle, |2\rangle\}$ basis, i.e. $\{|g\rangle_a |N+1\rangle_{\omega_\ell}, |e\rangle_a |N\rangle_{\omega_\ell}\}$

$$H = \hbar \begin{pmatrix} \omega_{eg} + \delta & \Omega/2 \\ \Omega^*/2 & \omega_{eg} \end{pmatrix}$$

subtract "$\hbar\omega_\ell N$" & E_g from diagonal energies ($\delta = \omega_\ell - \omega_{eg}$) [it's just an energy offset]

$$\Longleftrightarrow H = \hbar \begin{array}{c c} & \begin{array}{cc} |1\rangle & |2\rangle \end{array} \\ \begin{array}{c} \langle 1| \\ \langle 2| \end{array} & \begin{pmatrix} \delta & \Omega/2 \\ \Omega^*/2 & 0 \end{pmatrix} \end{array}$$

subtract "$\hbar\omega_{eg}$" from diagonal terms.

Note: A rigorous derivation requires going into the rotating frame of the atom (not an approximation).

AC Stark shift

Let's solve for the eigenenergies of the Hamiltonian

$$\det\{H - \lambda I\} = 0 \iff \begin{vmatrix} \delta - \lambda & \Omega/2 \\ \Omega^*/2 & 0 - \lambda \end{vmatrix} = 0$$

$$\iff (\delta - \lambda)(-\lambda) - \frac{|\Omega|^2}{4} = 0$$

$$\iff \lambda^2 - \delta\lambda - \frac{|\Omega|^2}{4} = 0$$

$$\Rightarrow \lambda_\pm = \frac{\delta \pm \sqrt{\delta^2 + |\Omega|^2}}{2} = \frac{E_\pm}{\hbar}$$

for $\underline{\delta = 0}$: $\boxed{E_\pm = \pm \frac{\hbar|\Omega|}{2}}$ "on-resonance"

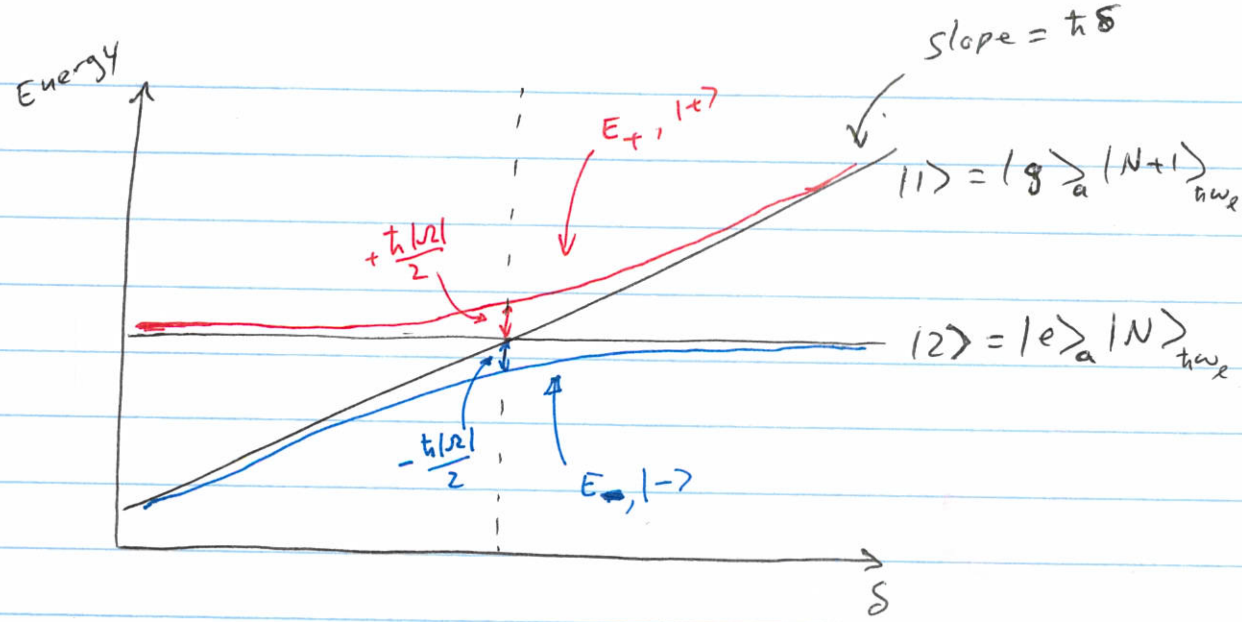
for $\underline{|\delta| \gg |\Omega|}$ "far off resonance"

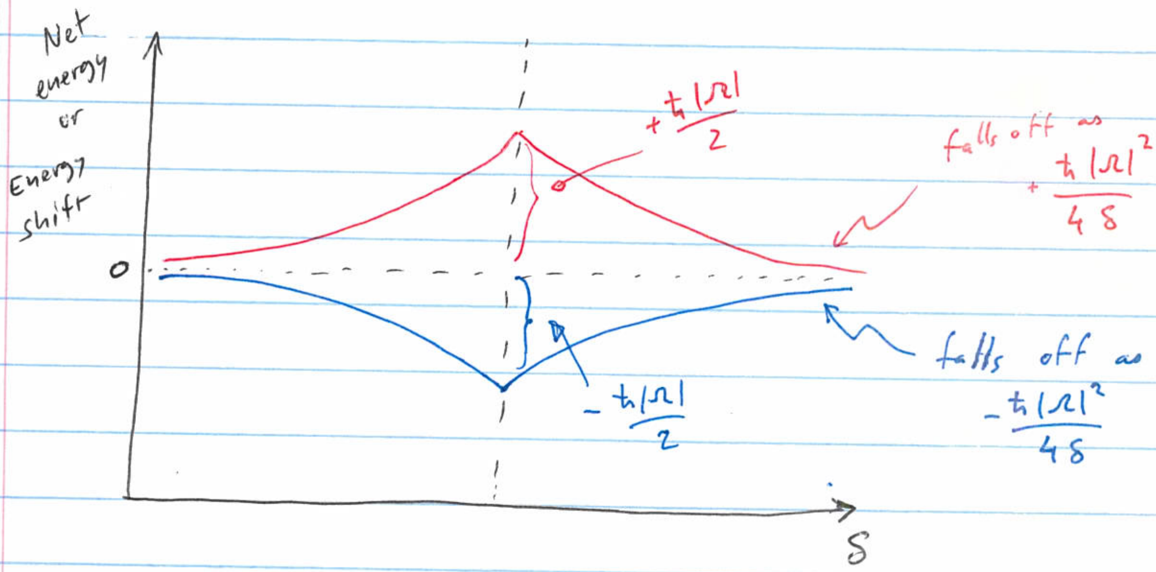
$$\sqrt{\delta^2 + |\Omega|^2} \simeq |\delta| + \frac{1}{2}\frac{|\Omega|^2}{|\delta|}$$

Thus $E_\pm = \hbar \dfrac{\delta \pm \left(|\delta| + \frac{1}{2}\frac{|\Omega|^2}{|\delta|}\right)}{2}$

$$\Rightarrow \begin{cases} E_+ = \boxed{\dfrac{\hbar(\delta + |\delta|)}{2}} + \dfrac{\hbar|\Omega|^2}{4|\delta|} \\ E_- = \boxed{\dfrac{\hbar(\delta - |\delta|)}{2}} - \dfrac{\hbar|\Omega|^2}{4|\delta|} \end{cases}$$

laser energy, does not depend on "Ω".





Next, let's find the eigenstates (or "dressed states")

$$\frac{E_+}{\hbar} = \lambda_+ = \frac{\delta + \Omega'}{2} \quad \text{where} \quad \Omega' = \sqrt{\delta^2 + |\Omega|^2}$$

$$\begin{bmatrix} \delta - \lambda_+ & \Omega/2 \\ \Omega^*/2 & 0 - \lambda_+ \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} \frac{\delta}{2} - \frac{\Omega'}{2} & \Omega/2 \\ \frac{\Omega}{2} & -\frac{\delta}{2} - \frac{\Omega'}{2} \end{bmatrix} \begin{array}{c} \\ \\ \end{array} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \begin{array}{l} |1\rangle \\ |2\rangle \end{array} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad |+\rangle = \frac{\Omega}{2}|1\rangle - \left(\frac{\delta}{2} - \frac{\Omega'}{2}\right)|2\rangle$$

with normalization, we can write $\boxed{|+\rangle = \cos\theta|1\rangle + \sin\theta|2\rangle}$
(for $\Omega \in \mathbb{R}$)

similarly, $\boxed{|-\rangle = -\sin\theta|1\rangle + \cos\theta|2\rangle}$

with $\cos\theta = \dfrac{\Omega}{\sqrt{\Omega^2 + (\Omega' - \delta)^2}}$ and $\sin\theta = \dfrac{\Omega' - \delta}{\sqrt{\Omega^2 + (\Omega' - \delta)^2}}$

<u>note</u>: $\cos 2\theta = \cos^2\theta - \sin^2\theta = -\dfrac{\delta}{\Omega'}$

<u>Example</u>: <u>On resonance</u>: $\delta = 0$, $\Omega' = \Omega$

$\cos\theta = \dfrac{1}{\sqrt{2}}$, $\sin\theta = \dfrac{1}{\sqrt{2}}$ $\Rightarrow \cos 2\theta = 0$

$\Rightarrow \theta = \pi/4$

$$\Rightarrow \begin{cases} |+\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle \\ |-\rangle = -\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle \end{cases}$$

These are the eigenstates of the atom-photon system, but we perform measurements in the $\{|1\rangle, |2\rangle\}$ basis typically, so we observe oscillations at $\omega = \frac{\Delta E}{\hbar} = \Omega$, i.e. Rabi flopping.

The atom-light interaction mixes the $|1\rangle$ and $|2\rangle$ states and lifts their degeneracy.

For EM field off : $|+\rangle = |1\rangle$

$|-\rangle = |2\rangle$