Thursday, March 7, 2024

Spontaneous Emission

I_ Basic Physics

Recall the dressed atom Hamiltonian:

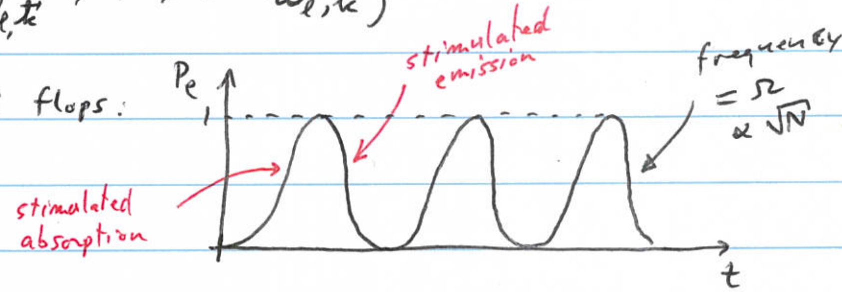
$$H = H_{atom} + H_{EM\,field} + H_{interaction}$$

$$= \begin{pmatrix} E_g & 0 \\ 0 & E_e \end{pmatrix} + \hbar\omega_{\ell,\vec{k}} \begin{pmatrix} N+1 & 0 \\ 0 & N \end{pmatrix} + \hbar \begin{pmatrix} 0 & \Omega/2 \\ \Omega^*/2 & 0 \end{pmatrix}$$

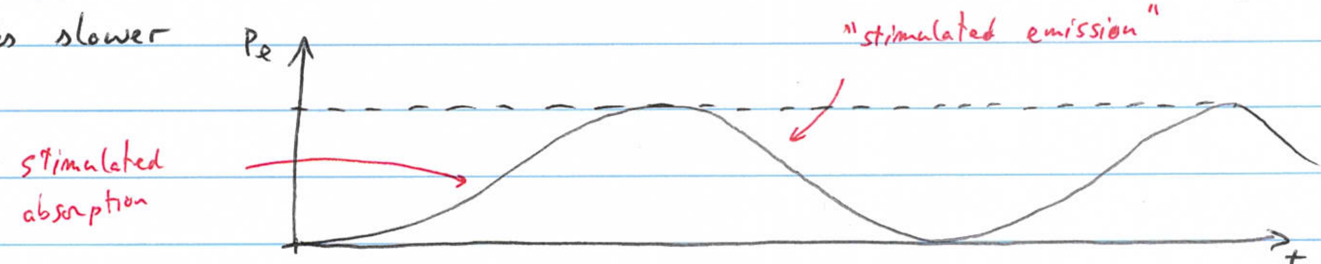
note: $\Omega \to \Omega(\sqrt{N})$

basis: $\left\{ |g\rangle|N+1\rangle_{\omega_\ell,\vec{k}},\ |e\rangle|N\rangle_{\omega_\ell,\vec{k}} \right\}$

Atom-field system Rabi flops:



suppose that we have a $N=1$ laser, then the Rabi-flopping is slower



photon vacuum state

The basis is $\left\{ |g\rangle|1\rangle_{\omega_\ell,\vec{k}},\ \underbrace{|e\rangle|0\rangle_{\omega_\ell,\vec{k}}} \right\}$

energetically, we can include $|e\rangle|0\rangle_{\omega_\ell,\vec{k}'}$ as well

So the basis is really

$$\left\{ |g\rangle|1\rangle_{\omega_\ell, \vec{k}},\ |e\rangle|0\rangle_{\omega_\ell, \vec{k}},\ \underbrace{|e\rangle|0\rangle_{\omega_\ell, \vec{k}' \neq \vec{k}}}_{\text{infinite number}},\ \dots \right.$$

during "stimulated emission" photon could go to $\vec{k} \neq \vec{k}'$

$$\left. \dots\ |g\rangle|1\rangle_{\omega_\ell, \vec{k}'} \right\}$$

II) Spontaneous Emission

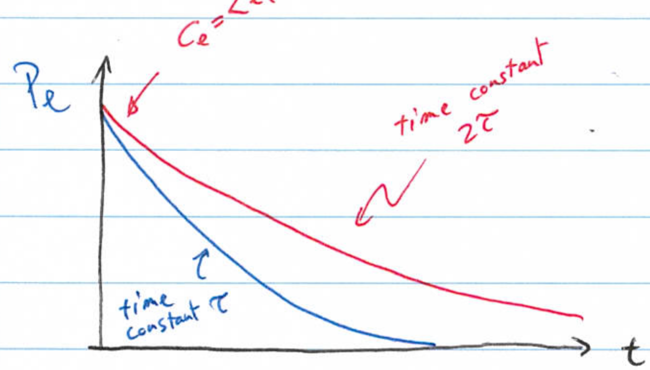
In other words, an excited atom can undergo "stimulated emission" by the photon vacuum in any direction.

- An atom in an excited state will tend to decay into a superposition of many $|g\rangle|1\rangle$ "one" photon states (all pointing in different directions).
- The more states into which an atom can "Rabi flop" or leak into, the faster the decay.

↳ The decay is exponential with time constant τ. For a 2-level atom

$$\tau = \frac{1}{\gamma} = \frac{3\pi \varepsilon_0 \hbar c^3}{|\langle e|R|g\rangle|^2\, \omega_{eg}^3} \simeq 27 \text{ ns for Rb}$$

density of photon $\vec{k}'$ states scales like ω_{eg}^3



$$\frac{dP_e}{dt} = -\gamma P_e$$

$$\frac{dC_e}{dt} = -\frac{\gamma}{2} C_e$$

III Derivation [originally derived by Wigner & Weisskopf]

[method: show that decay is exponential, i.e. $\frac{d C_e}{dt} = -\frac{\gamma}{2} C_e$

Dressed atom wavefunction for decaying excited states

$$|\psi(t)\rangle = c_e(t)|e\rangle + \sum_{\gamma \text{ photon states}} C_{g,\gamma}(t)\, \underbrace{e^{-i\delta_\gamma t}}\, |g\rangle|1\rangle_\gamma$$

with $\delta_\gamma = \omega_\gamma - \omega_{eg}$

we would like to write $\delta_\gamma = 0$ to conserve energy, but since the excited state exists for a finite time → then there ~~exists~~ is an energy spread.

The schrodinger equation becomes:

infinite number

$$i\hbar \frac{d}{dt}\begin{pmatrix} C_{g,\gamma_1} e^{-i\delta_1 t} \\ C_{g,\gamma_2} e^{-i\delta_2 t} \\ \vdots \\ C_{g,\gamma_N} e^{-i\delta_N t} \\ C_e \end{pmatrix} = \hbar \begin{pmatrix} \delta_1 & & & 0 & \frac{\Omega_{\gamma_1}}{2} \\ & \delta_2 & & & \frac{\Omega_{\gamma_2}}{2} \\ & & \ddots & & \vdots \\ 0 & & & \delta_N & \frac{\Omega_{\gamma_N}}{2} \\ \frac{\Omega_{\gamma_1}^*}{2} & \frac{\Omega_{\gamma_2}^*}{2} & \cdots & \frac{\Omega_{\gamma_N}}{2} & 0 \end{pmatrix} \begin{pmatrix} C_{g,\gamma_1} e^{-i\delta_1 t} \\ C_{g,\gamma_2} e^{-i\delta_2 t} \\ \vdots \\ C_{g,\gamma_N} e^{-i\delta_N t} \\ C_e \end{pmatrix}$$

$$\Rightarrow \begin{cases} i\dfrac{d}{dt}C_e(t) = \displaystyle\sum_{\gamma\,\text{states}} C_{g,\gamma}(t)\,\frac{\Omega_\gamma^*}{2}\,e^{-i\delta_\gamma t} & (1) \\[2ex] i\dfrac{d}{dt}C_{g,\gamma}(t)\cdot e^{-i\delta_\gamma t} + \cancel{\delta_\gamma\cdot C_{g,\gamma}(t)\,e^{-i\delta_\gamma t}} = \cancel{\delta_\gamma C_{g,\gamma}\,e^{-i\delta_\gamma t}} + \dfrac{\Omega_\gamma}{2}C_e(t) \end{cases}$$

$$\hookrightarrow\ i\frac{d}{dt}C_{g,\gamma}(t) = \frac{\Omega_\gamma}{2}\cdot C_e(t)\,e^{+i\delta_\gamma t} \qquad (2)$$

infinite number of equations (1 for each γ state)

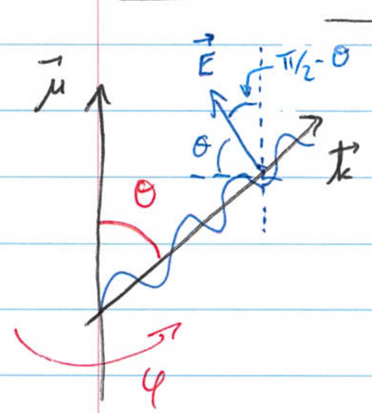
next, integrate (2)

$$C_{g,\gamma}(t) = -i\,\frac{\Omega_\gamma}{2}\int_0^t e^{i\delta_\gamma t'}\,C_e(t')\,dt'$$

substitute into (1):

$$\frac{d}{dt}C_e(t) = \underbrace{-\sum_\gamma \frac{|\Omega_\gamma|^2}{4}\int_0^t e^{-i\delta_\gamma(t-t')}}_{\text{evaluate this part in the hope of pulling } C_e(t') \text{ term out of integral.}}\,C_e(t')\,dt'$$

Recall: $\Omega_\gamma = \dfrac{\langle g|H_{int}|e\rangle}{\hbar} = \dfrac{\langle g|-e\vec{R}\cdot\vec{E}_\gamma|e\rangle}{\hbar}$

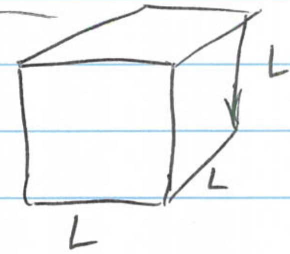


$$= \underbrace{-e\langle g|\vec{R}|e\rangle}_{\substack{\text{electric dipole}\\ \text{transition moment}\\ =\ \vec{\mu}_{eg}}}\cdot\frac{\vec{E}_\gamma}{\hbar} = -\vec{\mu}_{eg}\cdot\frac{\vec{E}_\gamma}{\hbar}$$

$$= -\frac{1}{\hbar}|\vec{\mu}_{eg}||\vec{E}_\gamma|\sin\theta$$

θ is in the direction of $\vec{k}$ not $\vec{E}$.

Cartoon quantization of the EM-field

Universe = Box with side L = 

$$\Rightarrow \quad \text{volume} = V = L^3$$

Energy of a photon in Box $= \hbar\omega_\gamma = \frac{1}{2}\int\left(\varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right) dV$

a little bit of algebra

$$\Rightarrow |\vec{E}_\gamma|_{RMS} = \sqrt{\frac{\hbar\omega_\gamma}{\varepsilon_0 V}}$$

So for Energy $= \frac{\hbar\omega_\gamma}{2}$ $\Rightarrow |\vec{E}_{\gamma, N=0}|_{RMS} = |\vec{E}_{vacuum}|_{RMS}$

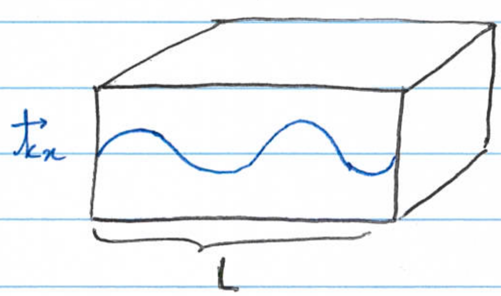
idea: vacuum has energy $\frac{\hbar\omega}{2}$

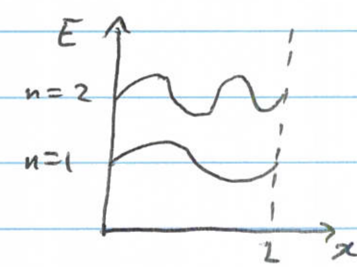
$$= \sqrt{\frac{\hbar\omega_\gamma}{2\varepsilon_0 V}}$$

$$\Rightarrow |\vec{E}_{vacuum}| = \sqrt{\frac{\hbar\omega_\gamma}{\varepsilon_0 V}} \equiv |\vec{E}_\gamma|$$

stopped here

Let's count the number of photon spatial modes in the box universe with periodic boundary conditions





$$\Rightarrow k_x = \frac{2\pi\, n_x}{L} \quad \Rightarrow \quad dn_x = \frac{L}{2\pi}\, dk_x$$

number of modes between k & k+dk

$$\Rightarrow dN = \left(\frac{L}{2\pi}\right)^3 dk^3 = dn_x dn_y dn_z$$

$\downarrow$ 2 polarizations

$$dN = 2 \frac{V}{(2\pi)^3} \underset{\downarrow\, \frac{\omega_\gamma}{c}}{k^2} \sin\theta \, \underset{\downarrow\, \frac{d\omega_\gamma}{c}}{dk} \, d\theta \, d\varphi$$

$$\Rightarrow \underbrace{dN_\gamma = 2 \frac{V}{(2\pi)^3} \frac{\omega^2}{c^3} \sin\theta \, d\omega \, d\theta \, d\varphi}_{\text{convert to integral}}$$

<u>Thus</u>: $\frac{d}{dt} c_e(t) = -\sum_{\gamma \text{ states}} \frac{|\Omega_\gamma|^2}{4} \int_0^t e^{-i\delta_\gamma(t-t')} c_e(t')\, dt'$

$$\frac{d}{dt} c_e(t) = -\int_0^\infty d\omega_\gamma \left\{ \frac{\omega_\gamma^2}{c^3} \frac{2V}{(2\pi)^3} \frac{|\mu_{eg}|^2 |E_\gamma|^2}{4\hbar^2} \right. \underbrace{\int_0^\pi d\theta \sin^3\theta}_{4/3} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}$$

$|E_\gamma|^2 \to \frac{\hbar\omega_\gamma}{\varepsilon_0 V}$; $\sin^2\theta$, $\sin\theta$

$\delta_\gamma = \omega_\gamma - \omega_{eg}$

$$\times \underbrace{\int_0^t e^{-i\delta_\gamma(t-t')}}_{\text{strongly peaked at } \begin{cases}\omega_\gamma = \omega_{eg} \\ t = t'\end{cases}} c_e(t')\, dt'$$

$\hookrightarrow \pi\, \delta(\delta_\gamma)\, \tilde{\delta}(t-t') - \underbrace{P\left[\frac{i}{\delta}\right]}_{\substack{\text{Lamb shift} \\ \text{(neglect)}}}$

$$\simeq -\frac{1}{2} \underbrace{\frac{\mu_{eg}^2 \omega_{eg}^3}{3\pi^2 \varepsilon_0 \hbar c^3}}_{\gamma = 1/\tau} c_e(t)$$

$\gamma = 1/\tau$ = spontaneous emission rate

$$\Rightarrow \boxed{\frac{d}{dt} c_e(t) = -\frac{1}{2}\gamma\, c_e(t)}$$

decay is exponential with time constant $= \frac{2}{\gamma}$