Tuesday, March 19, 2024

Spontaneous Emission (continued)

convert to integral?

$\delta_\gamma = \omega_\gamma - \omega_{eg}$

recall: $$\frac{d}{dt} C_e(t) = -\sum_\gamma \frac{|\Omega_\gamma|^2}{4} \int_0^t e^{-i\delta_\gamma (t-t')} C_e(t')\,dt'$$

photon states (ω_γ, k_γ) polarization

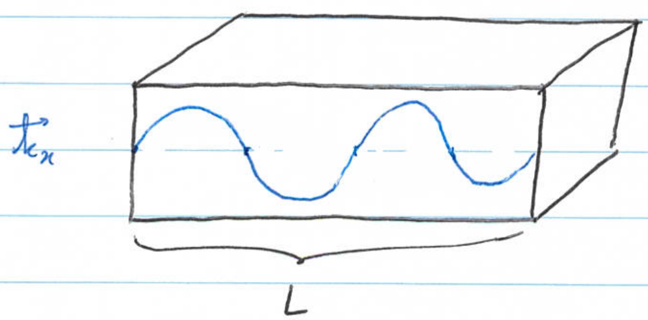
objective: rewrite above equation as $\frac{d}{dt} C_e(t) = -\underbrace{[\,?\,]}_{\gamma/2} C_e$

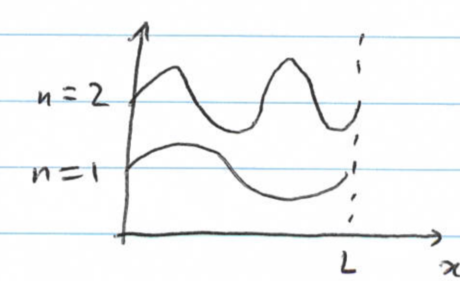
where γ = decay rate $= 1/\tau$

τ = decay time

Converting "$\sum_\gamma$" to an integral (sum over γ-photon states

Let's count the number of photon spatial modes in a box universe with periodic boundary conditions





$$\Rightarrow k_x = \frac{2\pi}{L} n_x \quad \Rightarrow \quad dn_x = \frac{L}{2\pi} dk_x$$

number of modes between k & $k + dk$

$$\Rightarrow \quad dN = \left(\frac{L}{2\pi}\right)^3 dk^3 = dn_x\, dn_y\, dn_z$$

$\downarrow$ 2 polarizations

$$dN = 2\,\frac{V}{(2\pi)^3}\,\underset{\substack{\uparrow \\ k = \frac{\omega_\gamma}{c}}}{k^2}\,\sin\theta\,\underset{\substack{\uparrow \\ \frac{d\omega_\gamma}{c}}}{dk}\,d\theta\,d\varphi$$

$$\Rightarrow \quad \underbrace{dN_\gamma = 2\,\frac{V}{(2\pi)^3}\,\frac{\omega_\gamma^2}{c^3}\,\sin\theta\,d\omega_\gamma\,d\theta\,d\varphi}_{\text{convert to integral}}$$

Thus: $$\frac{d}{dt} C_e(t) = -\sum_{\substack{\gamma \\ \text{states}}} \frac{|\Omega_\gamma|^2}{4} \int_0^t e^{-i\delta_\gamma (t-t')}\, C_e(t')\,dt'$$

recall: $\Omega_\gamma = -\frac{1}{\hbar}\,|\mu_{eg}|\,\underbrace{|\vec{E}_\gamma|}_{\sqrt{\frac{\hbar\omega_\gamma}{\varepsilon_0 V}}}\sin\theta \Rightarrow |\Omega_\gamma|^2 = \frac{|\mu_{eg}|^2}{\hbar}\,\frac{\omega_\gamma}{\varepsilon_0 V}\,\sin^2\theta$

$$\Rightarrow \frac{d}{dt} C_e(t) = -\int_0^\infty d\omega_\gamma \left\{ \frac{2V}{(2\pi)^3}\,\frac{\omega_\gamma^2}{c^3}\,\frac{|\mu_{eg}|^2}{4\hbar}\,\frac{\omega_\gamma}{\varepsilon_0 V} \underbrace{\int_0^\pi d\theta \sin^3\theta}_{4/3} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \right.$$

$$\left. \times \underbrace{\int_0^t e^{-i\delta_\gamma (t-t')}}_{\text{strongly peaked at } \begin{cases} \omega_\gamma = \omega_{eg} \\ t = t' \end{cases}} C_e(t')\,dt' \right\}$$

$\delta_\gamma = \omega_\gamma - \omega_{eg}$

$$\rightarrow \pi\,\delta(\delta_\gamma)\,\tilde{\delta}(t-t') - \underbrace{P\left[1/\delta\right]}_{\text{Lamb shift (neglect)}}$$

$$\Rightarrow \frac{d}{dt} c_e(t) = - \frac{\cancel{2}\cancel{V}}{(2\pi)^{\cancel{3}}} \frac{\omega_{eg}^3}{c^3} \frac{\mu_{eg}^2}{\cancel{4}\hbar} \frac{1}{\varepsilon_0 \cancel{V}} \frac{\cancel{4}}{3} \cancel{2\pi}\,\cancel{\pi}\, c_e(t)$$

$$= -\frac{1}{2} \underbrace{\boxed{\frac{\mu_{eg}^2 \omega_{eg}^3}{3\pi \varepsilon_0 \hbar c^3}}}_{\gamma = 1/\tau} c_e(t)$$

$= \text{spontaneous emission rate}$

$$\Rightarrow \boxed{\frac{d}{dt} c_e(t) = -\frac{1}{2}\gamma\, c_e(t)}$$

decay of c_e is exponential with time constant $= \frac{2}{\gamma}$

In "Schrodinger format", the equation becomes

$$i\hbar \frac{d}{dt} c_e(t) = -i\hbar \frac{\gamma}{2} c_e(t)$$

We would like to write the Schrodinger equation of the dressed atom with spontaneous emission as:

$$i\hbar \frac{d}{dt}\begin{pmatrix} c_g(t) \\ c_e(t) \end{pmatrix} = \hbar \begin{pmatrix} \delta & \Omega/2 \\ \Omega^*/2 & 0 \end{pmatrix}\begin{pmatrix} c_g(t) \\ c_e(t) \end{pmatrix} - \begin{pmatrix} 0 \;(?) \\ i\hbar \frac{\gamma}{2} c_e(t) \end{pmatrix}$$

We are not sure what to put here because as spontaneous emission depletes the excited state $|e\rangle$, atoms go into the ground state $|g\rangle$, but not coherently [the environment chooses a direction for the emitted photon]

"environment performs a measurement"

$$\Rightarrow i\hbar \frac{d}{dt}\begin{pmatrix} c_g(t) \\ c_e(t) \end{pmatrix} = \hbar \underbrace{\begin{pmatrix} \delta & \Omega/2 \\ \Omega^*/2 & -i\frac{\gamma}{2} \end{pmatrix}}_{H} \begin{pmatrix} c_g(t) \\ c_e(t) \end{pmatrix}$$

imaginary term
↳ non-Hermitian
⇒ probability is not conserved!!

⚠ The Schrodinger equation can only handle coherent superpositions of states, not statistical mixtures.

Two methods for dealing with statistical mixtures:

- Density matrix
- Monte Carlo wavefunctions
  or
  "Quantum trajectories"

Density Matrix

(Von Neumann, Landau, Bloch)

definition: The density matrix ρ for a system with wavefunction $|\psi\rangle$ is given by

$$\boxed{\rho = |\psi\rangle\langle\psi|}$$

If $|\psi\rangle = c_g|g\rangle + c_e|e\rangle$, then

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} c_g c_g^* |g\rangle\langle g| & c_g c_e^* |g\rangle\langle e| \\ c_e c_g^* |e\rangle\langle g| & c_e c_e^* |e\rangle\langle e| \end{pmatrix}$$

For a population in a statistical mixture of states $|\psi_i\rangle$ with probabilities p_i, then

$$\boxed{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|}$$

← allows for quantum superpositions and statistical mixtures simultaneously.

Properties:

Expectation value: $\langle A\rangle = \langle\psi|A|\psi\rangle = \mathrm{Tr}(\rho A)$

time evolution: $\boxed{i\hbar \frac{d}{dt}\rho = [H, \rho]}$

notation:

$$\rho = \begin{bmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{bmatrix} = \begin{bmatrix} c_g c_g^* & c_g c_e^* \\ c_e c_g^* & c_e c_e^* \end{bmatrix}$$

Resonance Fluorescence – optical Bloch Equations

The great advantage of the density matrix approach is that the decoherence that accompagnies spontaneous emission can be added in easily

coherent evolution: $i\hbar \frac{d}{dt}\rho = [H, \rho]$

$$\text{with} \qquad H = \begin{bmatrix} E_g & \hbar\Omega \cos(\omega_\ell t) \\ \hbar\Omega^* \cos(\omega_\ell t) & E_e \end{bmatrix}$$

(don't forget RWA)

$$\text{or} \qquad H_{dressed} = \begin{pmatrix} \delta & \Omega/2 \\ \Omega^*/2 & 0 \end{pmatrix}$$

$$\begin{cases} \frac{d}{dt}\rho_{gg} = \frac{-i}{\hbar}[H, \rho]_{gg} + \gamma \rho_{ee} \\ \frac{d}{dt}\rho_{ee} = \frac{-i}{\hbar}[H, \rho]_{ee} - \gamma \rho_{ee} \\ \frac{d}{dt}\rho_{eg} = \frac{-i}{\hbar}[H, \rho]_{eg} - \frac{1}{2}\gamma \rho_{eg} \\ \frac{d}{dt}\rho_{ge} = \frac{-i}{\hbar}[H, \rho]_{ge} - \frac{1}{2}\gamma \rho_{ge} \end{cases}$$

$\Rightarrow$ solve for $\rho_{gg}, \rho_{ee}, \rho_{eg}, \rho_{ge}$