

### Problem set #3

#### 1. Transverse E-field transformation – alternate derivation

Use the Lorentz transformation properties of the 4-potential and the 4-divergence/gradient operator to directly prove that

$$\vec{E}'_{\perp} = \gamma(\vec{E} + \vec{v} \times \vec{B})_{\perp}$$

#### 2. Magnetic dipole in motion

a) A magnetic dipole moves with constant velocity  $\vec{v}$  as viewed in the lab frame. Find the vector potential  $\vec{A}$  and scalar potential  $V$  in the lab frame.

**Hint:** It may be useful to employ the vector  $\vec{R} = \vec{r} - \vec{v}t$ .

b) Go to the non-relativistic limit ( $v \ll c$ ) and show that the moving magnetic dipole possesses both a magnetic dipole moment and an electric dipole moment.

#### 3. Jackson problem 11.19

#### 4. Relativistic charged particle in a constant electric field

A point charge  $q$  with mass  $m$  moves in a uniform electric field  $\vec{E} = E\hat{z}$ . The initial energy (kinetic energy plus rest energy), relativistic momentum, and velocity are  $U_0$ ,  $p_0$ , and  $v_0\hat{y}$ , respectively. Find  $\vec{r}(t)$  and show that eliminating  $t$  gives the particle trajectory:

$$z = \frac{U_0}{qE} \cosh\left(\frac{qEy}{cp_0}\right)$$

Check the non-relativistic limit.

#### 1. Lagrangian and Hamiltonian of a non-relativistic charged particle

While the electromagnetic force on a charged particle cannot be derived exclusively from a scalar potential (as required for standard Lagrangian mechanics), in this problem you will show that there exists a Lagrangian functional that produces the correct equations of motion for a charged particle in an arbitrary electromagnetic field.

**a)** Write down the Lorentz force law for charged particle in an electric field  $\vec{E}(\vec{r}, t)$  and magnetic field  $\vec{B}(\vec{r}, t)$ . Write down the Lorentz force law in terms of the electric potential  $V(\vec{r}, t)$  and the vector potential  $\vec{A}(\vec{r}, t)$ .

**b)** Show that the Lagrangian functional  $L$  below produces the Lorentz force law for a particle of charge  $q$  and mass  $m$  when it is used in conjunction with the Lagrange equation of motion (i.e. the equations that follow from the principle of least action):

$$L = \frac{1}{2} m \dot{\vec{r}}^2 + q \dot{\vec{r}} \cdot \vec{A}(\vec{r}, t) - qV(\vec{r}, t)$$

**c)** Derive an expression for the canonical momentum  $\vec{p}$ , and show that the Hamiltonian  $H$  for the charged particle can be written as

$$H = \frac{1}{2m} (\vec{p} - q\vec{A}(\vec{r}, t))^2 + qV(\vec{r}, t).$$