

Thursday, January 18, 2018

(Zangwill, chpt. 2, chpt 15.3)
(Jackson, chpt 6.1-6.3)

Maxwell's Equations for \vec{E} & \vec{B} and V and \vec{A}
(Equations of motion for ...)

electric & magnetic fields

scalar & vector potentials

For \vec{E} & \vec{B} : (in vacuum)

1) $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ ← charge density
Gauss's Law

2) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0$ ← Faraday's Law

3) $\vec{\nabla} \cdot \vec{B} = 0$ ← no magnetic monopoles law

4) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$ ← current density
Ampère's "improved" law

additional laws:

5) $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$ ← local conservation of charge
(continuity equation for charge)

this can be derived from Maxwell 1) & 4)

$\vec{E} \rightarrow \vec{E}(\vec{r}, t)$ in V/m
 $\vec{B} \rightarrow \vec{B}(\vec{r}, t)$ in Tesla (1 Tesla = 10^4 Gauss)

$\rho \rightarrow \rho(\vec{r}, t)$ in $\frac{C}{m^3}$
 $\vec{J} \rightarrow \vec{J}(\vec{r}, t)$ in $\frac{A}{m^2}$

c) $\vec{F} = \text{force on } \overset{\text{test}}{\text{charge}} = q[\vec{E} + \vec{v} \times \vec{B}]$ ← Lorentz force law
↑ measurable \vec{E} & \vec{B} are real quantities

Equations 1-4 (Maxwell's equations) fully determine \vec{E} & \vec{B} from $\rho(\vec{r}, t)$ and $\vec{J}(\vec{r}, t)$ (+ boundary conditions).

$$\left. \begin{aligned} \vec{E} &= (E_x, E_y, E_z) \\ \vec{B} &= (B_x, B_y, B_z) \end{aligned} \right\} 6 \text{ unknowns}$$

Maxwell's equations = $1 + 3 + 1 + 3 = 8$ equations
 2 equations are redundant

Some of these are interdependent \rightarrow there are not truly 6 unknowns

\hookrightarrow Q: how many true unknowns are there?

step back to statics:

Electrostatics:
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}') (\hat{r}-\hat{r}')}{|\vec{r}-\vec{r}'|^2}$$

or even earlier:
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

3 variables
 \downarrow
 1 variable
 $\vec{E} = -\vec{\nabla}V$
 follows from
 $\vec{\nabla} \times \vec{E} = 0$

Magnetostatics:
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times (\hat{r}-\hat{r}')}{|\vec{r}-\vec{r}'|^2}$$

or a little easier:
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r'$$

3 variables
 \downarrow
 $\vec{B} = \vec{\nabla} \times \vec{A}$
 follows from
 $\vec{\nabla} \cdot \vec{B} = 0$

With potentials there are only 4 unknowns (statics)

"Maxwell's Equations" for $V \neq \vec{A}$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{thus} \quad \boxed{\vec{B} = \vec{\nabla} \times \vec{A}} \quad (i)$$

↑ equivalent to Maxwell 3)

problem: $\vec{\nabla} \times \vec{E} \neq 0$ so $\vec{E} \neq -\vec{\nabla} V$ there is no scalar potential?

However: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Leftrightarrow \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

$$\Rightarrow \vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} = 0$$

$$\Rightarrow \boxed{\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0}$$

\Rightarrow there should be a " $V(\vec{r}, t)$ " such $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V$

$$\Leftrightarrow \boxed{\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}} \quad (ii)$$

equivalent to Maxwell 2) & 3)

Equations of motion for $V \neq \vec{A}$ (i.e. eliminate $\vec{E} \neq \vec{B}$)

$$(ii) \rightarrow 1) \Rightarrow \vec{\nabla} \cdot \left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \right) = \frac{\rho}{\epsilon_0}$$

$$\Leftrightarrow \boxed{\vec{\nabla}^2 V + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = -\frac{\rho}{\epsilon_0}} \quad (iii)$$

(i) & (iii) \rightarrow 4)

$$\Rightarrow \underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{A})}_{\vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \right)$$

$$\Leftrightarrow \left(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \nabla \left(\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \vec{J}(\vec{r}, t) \quad (iv)$$

looks like a wave equation

Equations (iii) & (iv) \rightarrow $\begin{cases} 4 \text{ equations} \\ 4 \text{ unknowns} \end{cases}$

\rightarrow "simpler" than Maxwell's equations

not so simple

Gauge Transformations (Gauge symmetry)

Equations (3) & (4) do not fully determine \vec{A} & V .

Also multiple \vec{A} & V give the same \vec{E} & \vec{B} .

If we do the following transformation,

$$\begin{cases} \vec{A}' = \vec{A} + \vec{\alpha}(\vec{r}, t) \\ V' = V + \beta(\vec{r}, t) \end{cases}$$

then what are the conditions on $\vec{\alpha}$ and β so that \vec{E} & \vec{B} remain unchanged?

If we want $\vec{B} = \nabla \times \vec{A} = \nabla \times \vec{A}'$ then we must have

$$\nabla \times \vec{\alpha} = 0$$

\Rightarrow there must be a $\lambda(\vec{r}, t)$ such that

$$\vec{\alpha}(\vec{r}, t) = \nabla \lambda(\vec{r}, t)$$

(in analogy with \vec{E} : $\nabla \times \vec{E} = 0 \Rightarrow \vec{E} = -\nabla V$)

Also, we want $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} = -\nabla V' - \frac{\partial \vec{A}'}{\partial t}$ | thus

$$= -\nabla V - \frac{\partial \vec{A}}{\partial t} - \nabla \left(\beta + \frac{\partial \lambda}{\partial t} \right)$$

require = 0

We require $\vec{\nabla} \left(\beta + \frac{\partial \lambda}{\partial t} \right) = 0$

$$\Rightarrow \beta + \frac{\partial \lambda}{\partial t} = k(t)$$

↑ constant in space

$$\Rightarrow \beta = -\frac{\partial \lambda}{\partial t} + k(t) = \frac{\partial}{\partial t} \left(-\lambda + \int k(t) dt \right)$$

rename " $-\lambda$ "

$$\Rightarrow \beta = -\frac{\partial \lambda(\vec{r}, t)}{\partial t}$$

So \vec{E} & \vec{B} do not change so long as we perform the following gauge transformation simultaneously

$$\begin{aligned} \vec{A}' &= \vec{A} + \vec{\nabla} \lambda \\ V' &= V - \frac{\partial \lambda}{\partial t} \quad \text{"} + k(t) \text{"} \end{aligned}$$

i.e. you can take any function $\lambda(\vec{r}, t)$: $\left. \begin{array}{l} \text{add } \vec{\nabla} \lambda \text{ to } \vec{A} \\ \text{add } -\frac{\partial \lambda}{\partial t} \text{ to } V \end{array} \right\}$

↳ \vec{E} & \vec{B} remain unchanged!

Gauge symmetry is an example of a continuous symmetry

similar to changing the spatial / time origin⁴ of a system
or the reference frame for a system → physics is unchanged

A judicious choice of $\lambda(\vec{r}, t)$ can simplify equations (iii) & (iv).