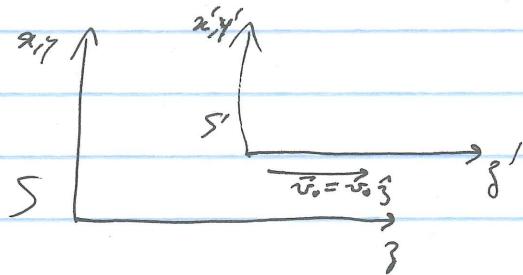


Thursday, January 25, 2018

Lorentz transformation



$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$\Delta (\vec{v}_0) \swarrow \text{time} \quad \nwarrow \text{column}$

Last time we also saw that you can think of the Lorentz transformation as a "rotation" in spacetime.

$\left\{ \begin{array}{l} \text{imaginary angle} \\ \text{imaginary time} \end{array} \right.$

Inverse Lorentz transformation

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & +\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

$\Delta(-\vec{v})$ ↑ line
↑ column

note: $\Delta(\vec{v})^{-1} = \Delta(-\vec{v})$ thus $\Delta(-\vec{v})\Delta(\vec{v}) = I$

↑
identity
matrix

Multiple Lorentz transformations

Two successive collinear Lorentz transformations are also a Lorentz transformation (unsurprising from a physics standpoint, this is also true for Galilean relativity)

$$\Delta(v_1 \hat{j}) \Delta(v_2 \hat{j}) = \Delta(v_3 \hat{j})$$

note: $v_3 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$



Important: $\Delta(v_1) \Delta(v_2) \neq \Delta(v_3)$

$$= \Delta(v_3) R(\phi)$$

$$= R(\phi') \Delta(\vec{v}_3')$$

(different from Galilean relativity)

↳ Leads to phenomena such as Thomas precession (atomic physics)
 & storage rings

Collinear Lorentz transformations form a group [Lorentz transformations on their own do not form a group]

- 1) Closure: $\Delta_a \Delta_b = \Delta_c$ | 3) Identity: $\Delta(o) \Delta_a = \Delta_a \Delta(o) = \Delta_a$
- 2) Associativity: $(\Delta_a \Delta_b) \Delta_c = \Delta_{ac} (\Delta_b \Delta_c)$ | 4) Inverse: $\Delta(v \hat{j}) \Delta(-v \hat{j}) = \Delta(o) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
 (property of matrix multiplication)

4-vectors

contravariant vectors: $a^\mu = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}$

covariant vectors: $a_\mu = (a_0, a_1, a_2, a_3)$
 $= (-a^0, a^1, a^2, a^3)$

Q: How do you transform a contravariant 4-vector from one ~~frame~~ reference frame to another?

A: $a^\mu = \sum_{\nu=0}^3 \Delta(\vec{v})^\mu_\nu a^\nu = \underbrace{\Delta(\vec{v})^\mu_\nu}_{\substack{\text{line} \\ \text{column}}} a^\nu$ Einstein notation
 implied summation over repeated indices

$$a^\mu = \Delta^\mu_\nu a^\nu = a^\nu \Delta^\mu_\nu$$

⚠ only sum over a top and a bottom index

$$\Leftrightarrow \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}$$

in S frame

in S' frame

note: $a^\mu = \Delta(-\vec{v})^\mu_\nu a^\nu$

$$\Delta(\vec{v})^\mu_\nu \Delta(-\vec{v})^\nu_\gamma = \delta^\mu_\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Scalar product

$$(\alpha)^2 = \alpha_\mu \alpha^\mu = -(\alpha^0)^2 + (\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2$$

$$\alpha b = \alpha_\mu b^\mu = -\alpha^0 b^0 + \alpha^1 b^1 + \alpha^2 b^2 + \alpha^3 b^3$$

The scalar product of two 4-vectors is Lorentz invariant

(i.e. the scalar product does not depend on your reference frame)

↳ this is one of the main reasons for using 4-vectors!

Q: How do covariant 4-vectors transform?

$$\alpha'_\mu = \Delta(-\vec{\omega})^\nu_\mu \alpha_\nu$$

$$\begin{aligned} (\alpha'_0, \alpha'_1, \alpha'_2, \alpha'_3) &= (\underbrace{\alpha_0, \alpha_1, \alpha_2, \alpha_3}_{(-\alpha^0, \alpha^1, \alpha^2, \alpha^3)}) \begin{bmatrix} \gamma & 0 & 0 & +\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \\ &= (-\alpha^0, \alpha^1, \alpha^2, \alpha^3) \end{aligned}$$

Proof of the Lorentz invariance of the scalar product:

$$\begin{aligned} \alpha' b' &= \alpha'_\mu b^\mu = \underbrace{\Delta(-\vec{\omega})^\nu_\mu}_{\delta_\mu^\nu} \alpha_\nu \underbrace{\Delta(\vec{\omega})^\mu_\nu}_{\delta_\nu^\mu} b^\nu \\ &= \delta_\mu^\nu \alpha_\nu b^\nu = \alpha_\mu b^\mu = ab \end{aligned}$$

The metric tensors $g^{\mu\nu}$ and $g_{\mu\nu}$ allow one to switch between contravariant and covariant 4-vector forms

$$a^\mu = g^{\mu\nu} a_\nu \quad \text{and} \quad b_\mu = g_{\mu\nu} b^\nu$$

For special relativity: $[g^{\mu\nu}] = [g_{\mu\nu}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Note: $ab = a^\mu b_\mu = g^{\mu\nu} a_\nu b_\mu = g_{\mu\nu} a^\mu b^\nu$

Note:

$$g^{\mu\nu} = g^{\nu\mu}$$

$$g^{\mu\nu} g_{\nu\gamma} = g^{\mu\gamma} = \delta_\gamma^\mu$$

"East coast" metric

alternative (i.e. Jackson)

$$\begin{bmatrix} 1 & & & \\ & -1 & 0 & 0 \\ 0 & & -1 & \\ & & & -1 \end{bmatrix}$$

$$\Rightarrow a_\mu = (a^0, -a^1, -a^2, -a^3)$$

Lorentz transformation matrix is the same $\Delta(\vec{v})^\mu_\nu$

Examples of 4-vectors:

4-position vector: $x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ \vec{r} \end{pmatrix}$

$$\begin{aligned} x^\mu x_\mu &= \text{invariant interval} = -c^2 t^2 + x^2 + y^2 + z^2 \\ &= -c^2 t'^2 + x'^2 + y'^2 + z'^2 \end{aligned}$$

proper time: if we record the 4-positions of a particle at 2 different times we can construct the invariant interval

$$dx^\mu dx_\mu \rightarrow dx^\mu dx_\mu = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

for the frame on the particle (s'): $dx'^\mu dx'_\mu$

$$= -c^2 dt'^2 + dx'^2 + dy'^2 + dz'^2$$

$$= 0$$

particle does
not move
in its frame

$$dx^\mu dx_\mu = dx'^\mu dx'_\mu \Leftrightarrow dt'^2 = dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2}$$

$$\Rightarrow \frac{dt'^2}{dt^2} = 1 - \frac{v^2}{c^2}$$

$$\Rightarrow dt' = \sqrt{1 - \frac{v^2}{c^2}} dt = \frac{1}{\gamma} dt$$

rename: $dt' \rightarrow d\tau$ = proper time (differential) \rightarrow time for clock on particle

$$= \frac{1}{\gamma} dt$$

↑ time differential in s (frame in which particle is moving)

$$\tau = \frac{t}{\gamma} = \text{proper time}$$

4-velocity: $v^\mu = \frac{dx^\mu}{dt} = \gamma \frac{dx'}{dt} = \left(\begin{array}{c} \gamma c \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{array} \right)$

⚠ $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ \vec{v} = velocity of particle in s frame

$$\left(\begin{array}{c} \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{array} \right)$$

but $\nabla \cdot (\vec{v}_0 = v_0 \hat{\vec{y}})$ with $\vec{v}_0 \neq \vec{v}$

$v_\mu v^\mu = -c^2$

note: $v_\mu v^\mu = c^2$
in "west coast" metric
i.e. Jackson

$$\text{4-acceleration: } \Gamma^\mu = \frac{d}{dt} v^\mu = \left(\gamma \dot{\gamma} c \right) \frac{d\gamma}{dt} + \left(\gamma \dot{\gamma} \vec{v} + \gamma^2 \vec{a} \right) \frac{d\vec{v}}{dt}$$

note: $\frac{d}{dt} v^\mu v_\mu = 2 v_\mu \frac{dv^\mu}{dt}$

$$(\Rightarrow) \frac{d}{dt} (-c^2) = 2 v_\mu \Gamma^\mu \Rightarrow \boxed{\Gamma^\mu v_\mu = 0}$$

proper acceleration of ~~the~~ of a particle P is the acceleration of the particle as measured in its instantaneous rest frame S'

f. rectilinear motion:

$$\begin{aligned} \alpha &= \frac{d\vec{dt}'}{dt'} \left[-\frac{d\vec{a}'_x}{dt'} \right]_{\vec{v}=v_x}^{v'_x=v_x} \\ \alpha &= \frac{d\vec{a}_x}{dt'} \left(1 - \frac{v_0 v_x}{c^2} \right) - (v_x \vec{v}) \cancel{\frac{d\vec{v}}{dt'}} - \left(\frac{v_0}{c^2} \right) \frac{d\vec{u}_x}{dt'} \\ &= \gamma^2 \frac{d\vec{u}_x}{dt'} \quad dt' = \frac{1}{\gamma} dt \\ &= \gamma^2 \frac{d\vec{u}_x}{dt} \quad \Rightarrow \quad \boxed{\alpha = \gamma^3 \frac{d\vec{u}_x}{dt} = \text{proper acceleration}} \end{aligned}$$

velocity of P in S' velocity of S' in S see problem set #2

$v'_x = v_x$

$\alpha'_x = \frac{u_x - v_0}{1 - \frac{v_0 u_x}{c^2}}$

$\alpha'_x = \frac{u_x - v_0}{1 - \frac{v_0 u_x}{c^2}} \quad v_0^2$

Stopped here

note: $\Gamma^\mu \Gamma_\mu = \alpha^2$

$$\begin{aligned} \text{4-momentum: } p^\mu &= m_0 v^\mu \\ &= \left(m_0 \gamma c \right) \frac{1}{m_0 \gamma \vec{v}} = \left(E/c \right) \frac{\vec{p}}{\gamma \vec{v}} \end{aligned}$$

rest mass 3-velocity 3-momentum relativistic kinetic energy + rest energy