PHYS 610: Electricity & Magnetism I
Due date: Thursday, April 26, 2018

Problem set #10

1. Conductor and dielectric

Consider a thin grounded conducting shell of radius *a* surrounded by a concentric spherical dielectric region of radius b with permittivity ε. There is vacuum for $r > b$. The entire system is subject to an applied external electric field $\vec{E} = E_0\hat{z}$. Calculate the potential everywhere.

2. The almost-concentric spherical capacitor

A spherical conducting shell centered at the origin has radius R_1 and is maintained at potential V_1. A second spherical conducting shell is maintained at potential V_2 has radius $R_2 > R_1$ but is somewhat off-center by a distance $\delta << R_1$ in the $+\hat{z}$ direction.

a) Calculate the capacitance of the system for $\delta = 0$.

b) To lowest order is δ, show that surface charge density induced on the inner shell is given by

$$\sigma(\theta) = \epsilon_0 \frac{R_1 R_2 (V_2 - V_1)}{R_2 - R_1}\left(\frac{1}{R_1^2} - \frac{3\delta}{R_2^3 - R_1^3}\cos\theta\right)$$

Note: It may be helpful to first show that the boundary of the outer spherical shell can be approximated as $r_2 = R_2 + \delta\cos\theta$.

c) Compute the total charge Q on the inner shell and the capacitance of the off-center capacitor to lowest order in δ.

d) To lowest order in δ, show that the inner shell experiences a total force

$$\vec{F} = -\frac{Q^2}{4\pi\epsilon_0}\frac{\delta}{R_2^3 - R_1^3}\hat{z}$$

Note: the force is given by the charge density and the electric field in the vicinity of the charge.

e) Integrate the force in (d) to find the capacitance of the system to second order in δ.

3. Dielectric sphere with free charge

A dielectric sphere (with dielectric constant κ) of radius R is filled with a uniform free charge density ρ_c.

a) Find the polarization $\vec{P}(\vec{r})$.

b) Calculate the total bound charge of the sphere (volume and surface). Explain the result briefly.

4. Static Magnetic Field Maxima and Minima

In this problem you will prove that a charge and current free region of space cannot have a maximum in the magnitude of the local magnetic field. While this fact may seem rather basic, it was not widely known until recently. A proof was published in the early 1980s by W. H. Wing. The theorem is a variation on Earnshaw's theorem for electrostatic potentials. The theorem is also attributed to Thomson.

Proof by contradiction

We place the origin of our coordinate system at the position of the suspected magnetic field maximum. The magnetic field maximum at the origin is denoted as $\vec{B}(0)$. As we move away from the origin, the magnetic field decreases by an amount $\delta\vec{B}(\vec{r})$, so that $\vec{B}(r) = \vec{B}(0) + \delta\vec{B}(\vec{r})$.

a) Show that the magnetic field must obey $\vec{B}(0)\cdot\delta\vec{B}(\vec{r}) < 0$.

b) If we choose the z-axis as the direction of the local magnetic maximum, then show that $\vec{B}(0)\cdot\delta\vec{B}(\vec{r}) = \vec{B}_z(0)\cdot\delta\vec{B}_z(\vec{r})$ and $\delta\vec{B}_z(\vec{r}) < 0$.

c) Use the vector relation $\nabla\times(\nabla\times\vec{B}) = \nabla(\nabla\cdot\vec{B}) - \nabla^2\vec{B}$ to show that $\nabla^2\vec{B} = 0$, $\nabla^2 B_z = 0$, and $\nabla^2\delta B_z = 0$.

d) Use Green's Theorem shown below to show that the average of δB_z over a sphere of radius *r* centered on the origin is equal to zero.

$$\int_V \left[\phi(\nabla^2\psi) - \psi(\nabla^2\phi)\right] d^3r = \int_S \left[\phi(\nabla\psi) - \psi(\nabla\phi)\right]\cdot d\vec{s}$$

hint: use $\psi = \delta B_z$ *and* $\phi = 1/r$.

e) Show that $\vec{B}(0)$ is not a magnetic field maximum.

f) Give an example of a current distribution which generates a local ***minimum*** in the magnetic field magnitude in a region of space free of currents or charges. Draw a sketch of the current distribution and the magnetic field minimum region.