

Tuesday, April 17, 2018

Capacitance

In any electrostatic system $\vec{E} \propto Q \}$ or ΔQ for
 $V \propto Q \}$ a neutral system

On a conductor, capacitance is the proportionality factor:

V is the same
everywhere on
conductor

depends only
on geometry
and materials

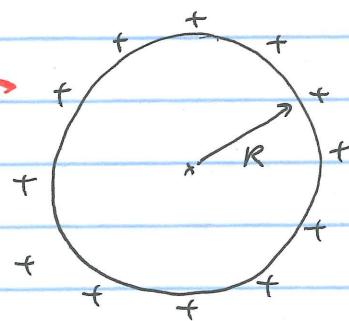
$$C = \frac{Q}{\Delta V}$$

= charge "stored"
per volt

Unit of capacitance = Farad
 $= [C]/[V]$

Example: Consider a conducting sphere of radius R with charge Q .

charge is on
surface and
uniformly distributed



Gauss's law gives for $r > R$: $4\pi r^2 E = \frac{Q}{\epsilon_0} \Rightarrow \vec{E} = \frac{Q}{4\pi \epsilon_0 r^2} \hat{r}$

$$\vec{E} = -\nabla V \Rightarrow V = - \int_{-\infty}^r \vec{E}(\vec{r}) \cdot d\vec{r}$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{1}{r} \quad \text{note: } V(r \rightarrow \infty) = 0$$

Q: How do you get this from multipole expansion?
 → A: only first term contributes
 (monopole)

V is continuous and $V(r < R) = \text{cst}$ thus

$$V(r < R) = \frac{Q}{4\pi\epsilon_0} \frac{1}{R}$$

$$\Rightarrow \frac{Q}{V} = 4\pi\epsilon_0 R \Rightarrow C = 4\pi\epsilon_0 R$$

or length

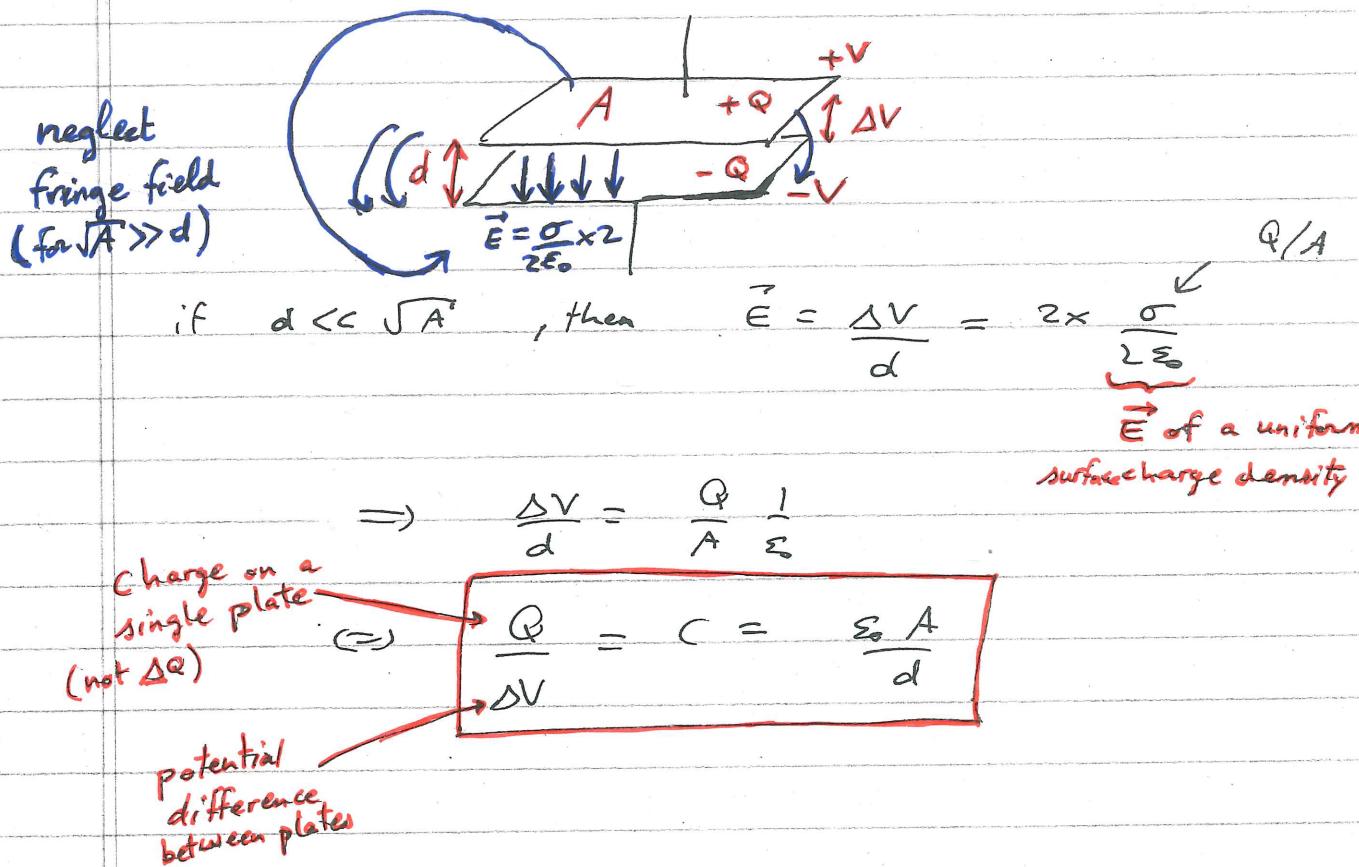
↳ since the charge on a conductor distributes itself uniformly over its surface, we expect the conductor surface A to be involved.

Note: $R = \sqrt{\frac{A}{4\pi}}$ $\Rightarrow C = \epsilon_0 \sqrt{4\pi A}$

generally, $C \approx \epsilon_0 \sqrt{4\pi A}$ for other shapes
 decent approximation

Capacitors are neutral two-terminal devices generally made up of at least 2 conductors

parallel plate capacitor:



Energy stored in a capacitor

If you start with an uncharged capacitor the work to move charge from one conductor/terminal to the other is

$$dW = \underbrace{V dq}_{q/C} \Rightarrow W = \int_0^Q \left(\frac{q}{C} \right) dq = \frac{1}{C} \frac{q^2}{2} \Big|_0^Q$$

$$= \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C \underbrace{\left(\frac{Q}{C} \right)^2}_V$$

$$= \frac{1}{2} CV^2$$

$$\Rightarrow U_{\text{capacitor}} = \frac{1}{2} C V^2 \quad \text{energy stored by a capacitor}$$

Q: Does a wire have a capacitance?

A: Yes, especially if it is resistive.



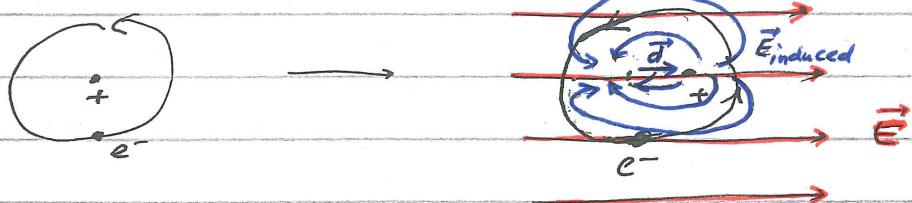
you can define/calculate capacitance with

$$\frac{\epsilon_0}{2} \int \vec{E}^2 d^3r = \frac{1}{2} C V^2$$

E-field \Rightarrow capacitance

Electrostatics in Matter

- 1) Consider a classical atom in an E -field :



the atom acquires a dipole \vec{p}
induced by the \vec{E} -field

$$\vec{p} = 19_e / \vec{d} = \propto \vec{E}$$

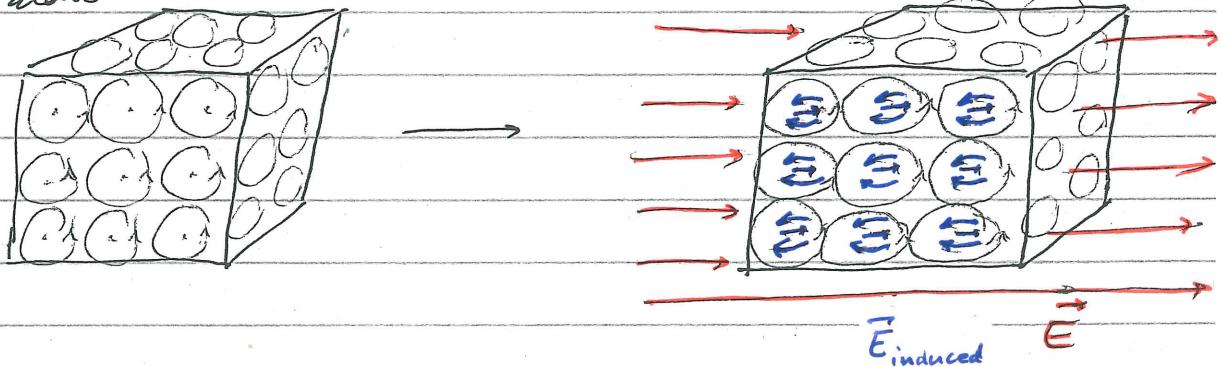
$$(\alpha > 0)$$

\downarrow
 $\alpha = \text{polarizability}$

Locally, the e^- and nuclear charges don't exactly cancel

→ the induced dipole \vec{p} produces its own electric field, \vec{E}_{induced} , which partially counteracts the applied E -field \vec{E} .

- 2) Consider a block of matter made up of similar-behaving atoms



→ you expect to polarize the material and so there should be an associated: \vec{E}_{induced}

→ dipole moment per unit volume: $\vec{p} = ? \propto \vec{E}$

We'll come back to this later

Thus \vec{E}_{total} is given by $\vec{E}_{\text{total}} = \vec{E}_{\text{applied}} + \vec{E}_{\text{induced}}$
 \vec{E}_{self}
 or
 $\vec{E}_{\text{internal}}$

→ you expect the charges in the material to not exactly cancel (i.e. you get local dipoles)

↳ volume bound charge ρ_b

surface bound charge σ_b

charges supplied by material

↳ total charge: $\rho_{\text{total}} = \rho_{\text{applied}} + \rho_b + \sigma_b$

$\rho_{\text{free}} = \text{free charge}$

= charge put there by the experimentalist

we also require

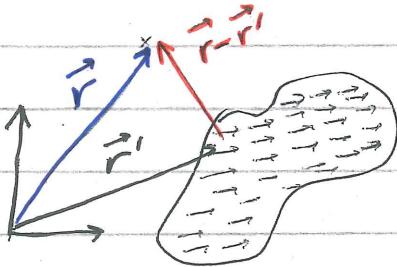
$$\int_{V \text{ of material}} \rho_b(\vec{r}) d^3r + \int_{S \text{ of material}} \sigma_b(\vec{r}) ds = 0$$

definition: $\vec{P} = \text{dipole moment per unit volume}$

↳ model polarized matter as a collection of dipoles
 \rightarrow (no quadrupoles)

Q: What is the relationship between \vec{P} , ρ_b , and σ_b ?

Consider the potential $V(\vec{r})$ produced by a polarized material



$$\text{for a single dipole } \vec{p} : V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2} \quad \vec{p}/m^3$$

$$\text{For a uniform continuous distribution of } \vec{p}'\text{'s : } V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\vec{P}(\vec{r}') \cdot (\hat{r}-\hat{r}') d^3 r'}{|\vec{r}-\vec{r}'|^2}$$

$$\text{However, we note that } \vec{\nabla}_{\vec{r}'} \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = \frac{\hat{r}-\hat{r}'}{|\vec{r}-\vec{r}'|^2}$$

a few lines of algebra

$$\text{Thus, } V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \vec{P}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) d^3 r'$$

$$\vec{\nabla} \cdot \left(\vec{P} \frac{1}{|\vec{r}-\vec{r}'|} \right) - \frac{1}{|\vec{r}-\vec{r}'|} \vec{\nabla} \cdot \vec{P}$$

$$= \frac{1}{4\pi\epsilon_0} \left\{ \int \vec{\nabla} \cdot \left(\vec{P} \frac{1}{|\vec{r}-\vec{r}'|} \right) d^3 r' + \int \frac{(-\vec{\nabla} \cdot \vec{P})}{|\vec{r}-\vec{r}'|} d^3 r' \right\}$$

divergence theorem

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\vec{P}(\vec{r}') \cdot \hat{n}' d\sigma'}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi\epsilon_0} \int \frac{(-\vec{\nabla}_{\vec{r}'} \cdot \vec{P}(\vec{r}'))}{|\vec{r}-\vec{r}'|} d^3 r'$$

surface charge distribution

volume charge distribution

standard form for the potential produced by a charge distribution

thus we often (or define)

$$\left\{ \begin{array}{l} \phi_b(\vec{r}) \equiv - \vec{\nabla} \cdot \vec{P}(\vec{r}) \\ \sigma_b(\vec{r}) \equiv \vec{P}(\vec{r}) \cdot \hat{n} \end{array} \right.$$

\hat{n} = normal unit vector
pointing out of surface.

example: calculate the potential outside a uniformly polarized dielectric sphere (radius = R) with polarization \vec{P} .

$$\vec{P} = P \hat{z}$$

$$\vec{\nabla} \cdot \vec{P} = 0 \Rightarrow \phi_b = 0$$

$$\vec{P} \cdot \hat{n} = P \hat{z} \cdot \hat{n} = P \cos \theta \Rightarrow \sigma_b(\vec{r}) = P \cos \theta$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{P \cos \theta' R^2 \sin \theta' d\theta' d\phi'}{|\vec{r} - \vec{r}'|} \quad \text{valid for } r > R$$

see Multipole

Y_m^m expansion

example in
previous lecture

$$= \frac{PR^3}{3\epsilon_0} \frac{\cos \theta}{r^2} \quad \text{for } r > R$$

it's a perfect dipole field.

stopped here

Gauss's Law and Electric Displacement

Gauss's law always applies, thus inside the material

$$\underbrace{\vec{\nabla} \cdot \vec{E}_{\text{total}}}_{\vec{E}} = \frac{l_{\text{total}}}{\epsilon_0} = \frac{1}{\epsilon_0} (l_{\text{free}} + l_b) \quad \uparrow \vec{J} \cdot \vec{P}$$

$$\Rightarrow \epsilon_0 \vec{\nabla} \cdot \vec{E} = l_{\text{free}} - \vec{\nabla} \cdot \vec{P}$$