

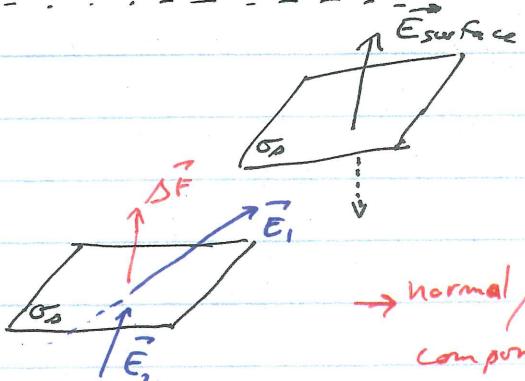
Friday, April 27, 2018

Matching conditions for \vec{B} across a surface current

Reminder for \vec{E} & V at a surface of charge:

$$\vec{E}_{\text{surface}} = \frac{\sigma_0}{2\epsilon_0} \hat{n}$$

$$\Delta \vec{E} \Big|_s = \vec{E}_1 \Big|_s - \vec{E}_2 \Big|_s = \frac{\sigma_0}{\epsilon_0} \hat{n}$$



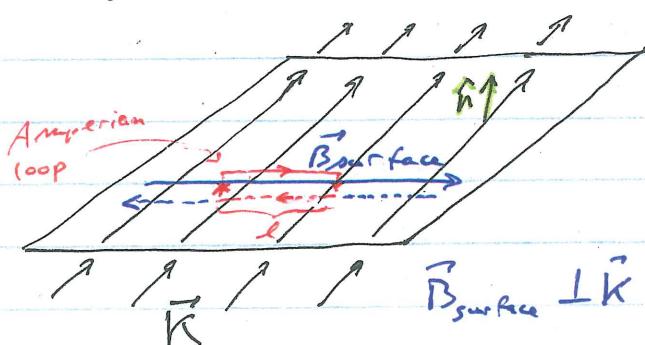
→ normal / perpendicular component is affected

→ parallel component is unaffected

$$\hookrightarrow \begin{cases} \Delta \vec{E} \Big|_s \cdot \hat{n}_i = \frac{\sigma_0}{\epsilon_0} \\ \Delta \vec{E} \Big|_s \times \hat{n}_i = 0 \end{cases} \Rightarrow \vec{E}_{1,\parallel} = \vec{E}_{2,\parallel}$$

Also: V is continuous across the interface

Consider a surface current \vec{K} :



$$\begin{aligned} \vec{K} &= \text{surface current} \\ &= \vec{J} \delta(z) \\ B_{\text{surface}} &\perp \vec{K} \quad = [A]/[m] \end{aligned}$$

Ampère's law: $\nabla \times \vec{B}_{\text{surface}} = \mu_0 \vec{J}$

$$\hookrightarrow \oint \vec{B}_{\text{surface}} \cdot d\vec{l} = \mu_0 I_{\text{enclosed}}$$

$$\Rightarrow B_{\text{surface},1} \cancel{l} + \cancel{\text{endcap}} + B_{\text{surface},2} \cancel{l} + \cancel{\text{endcap}} = \mu_0 K l$$

$$B_{\text{surface},1} = B_{\text{surface},2}$$

$$\Leftrightarrow$$

$$B_{\text{surface}} = \frac{\mu_0 K}{2}$$

$$\Rightarrow$$

$$B_{\text{surface}} = \frac{\mu_0}{2} \vec{K} \times \hat{n}$$

(above & below surface)

We now add an external B -field \vec{B}_{ext} : $\vec{B} = \vec{B}_{\text{surface}} + \vec{B}_{\text{ext}}$

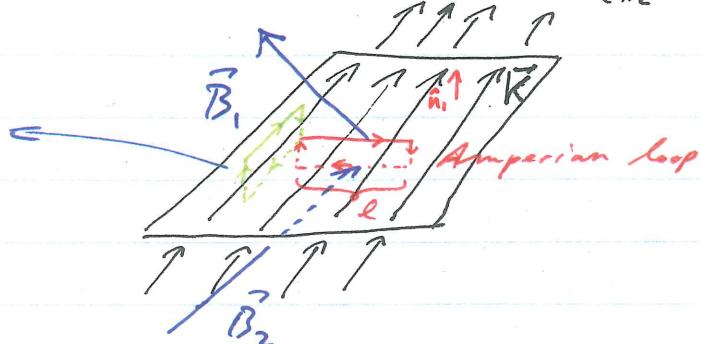
Ampère's law:

$$B_{1,\parallel} - B_{2,\parallel} = 0$$

$$\Rightarrow \oint \vec{B} \cdot \hat{k} = 0$$

$\parallel \rightarrow \parallel$ to surface

\parallel to \hat{k}



Ampère's law: $\oint \vec{B} \cdot d\vec{l} = \mu_0 K l$

$$\Rightarrow B_{1,\parallel}l - B_{2,\parallel}l = \mu_0 K l \Leftrightarrow B_{1,\parallel} - B_{2,\parallel} = \mu_0 K$$

$$\Rightarrow \Delta \vec{B} \cdot (\hat{k} \times \hat{n}_1) = \mu_0 K$$

$\parallel \rightarrow \parallel$ to surface
 \perp to \hat{k}

$$\Rightarrow \Delta \vec{B} = \vec{B}_1 - \vec{B}_2 = \mu_0 \hat{k} \times \hat{n}_1$$

$$\nabla \cdot \vec{B} = 0$$

"Gauss' law"

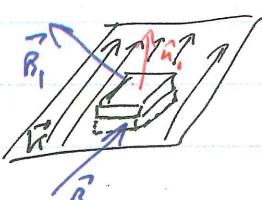
$$\oint \vec{B} \cdot d\vec{s} = 0$$

pill box

$$\vec{B}_1 \cdot \hat{n}_1 - \vec{B}_2 \cdot \hat{n}_1 = 0$$

$$\Rightarrow \Delta \vec{B} \cdot \hat{n}_1 = 0$$

surface
parallel



Qualitative result:

- perpendicular $\stackrel{\text{to surface}}{\parallel}$ component unaffected
- parallel $\stackrel{\text{to surface}}{\parallel}$ component affected
(but not the one \parallel to \hat{k})

what about \vec{A} ?

For the Coulomb gauge: $\nabla \cdot \vec{A} = 0 \rightarrow \Delta \vec{A} \cdot \hat{n}_1 = 0$

\hookrightarrow perpendicular component of \vec{A} is unaffected

for an amperian loop: $\oint \vec{A} \cdot d\vec{l} = \int_{\text{Loop}} (\vec{J} \times \vec{A}) \cdot d\vec{s}$

\vec{B}
Surface
of loop

$$= \int \vec{B} \cdot d\vec{s} \xrightarrow{\text{flux}} 0$$

reduce thickness of loop

$$\Rightarrow \Delta \vec{A} \Big|_{\parallel} = 0 \rightarrow \text{parallel components are unaffected}$$

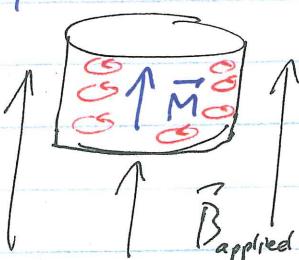
$$\Rightarrow \boxed{\Delta \vec{A} \Big|_S = 0} \Rightarrow \vec{A} \text{ is continuous across the interface (like } V)$$

Note: However $\frac{\partial A}{\partial n}$ is discontinuous.

B-fields in Matter [Griffiths, Zangwill] chpt 6 chpt 13

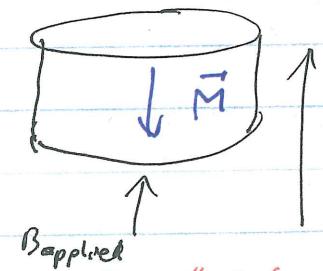
Basics: paramagnet

\vec{M} = "magnetization"
= magnetic moment per unit volume
generally from e^- spins



[attracted to] "high-field seeker"

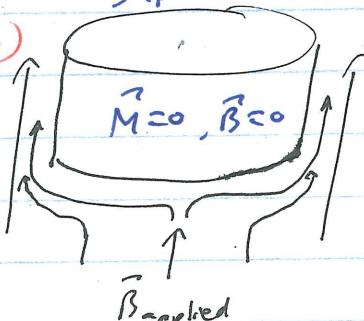
diamagnet



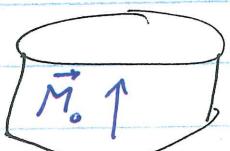
[repelled from] "low-field seeker"

but it can be useful to think of small current loops (classical description)

Superconductor



Ferrimagnet

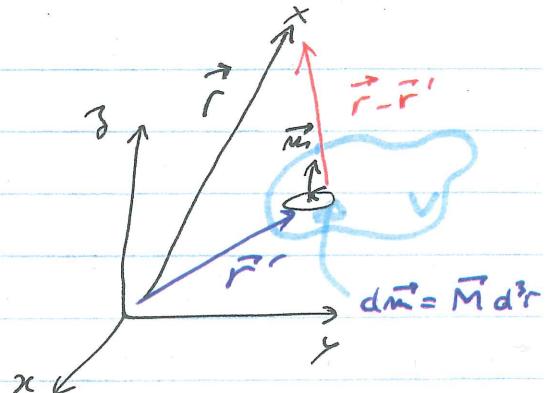


$B_{\text{applied}} = 0$

\vec{M} depends on history of \vec{B}_{applied}

Bound currents

$$\vec{A}_{\text{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times (\hat{\vec{r}} - \hat{\vec{r}'})}{|\vec{r} - \vec{r}'|^2}$$



$$\vec{A}_M(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{M}(\vec{r}') \times (\hat{\vec{r}} - \hat{\vec{r}'})}{|\vec{r} - \vec{r}'|^2} d^3 r'$$

however $\frac{(\hat{\vec{r}} - \hat{\vec{r}'})}{|\vec{r} - \vec{r}'|^2} = \vec{\nabla}_{r'} \frac{1}{|\vec{r} - \vec{r}'|}$ (see April 1st lecture)

$$\begin{aligned} \vec{M}(\vec{r}') \times \frac{(\hat{\vec{r}} - \hat{\vec{r}'})}{|\vec{r} - \vec{r}'|^2} &= \vec{M}(\vec{r}') \times \vec{\nabla}_{r'} \frac{1}{|\vec{r} - \vec{r}'|} \\ &= \frac{1}{|\vec{r} - \vec{r}'|} \vec{\nabla} \times \vec{M} - \vec{\nabla} \times \left(\frac{\vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) \end{aligned}$$

$$\Rightarrow \vec{A}_M(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{\nabla} \times \vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3 r' - \frac{\mu_0}{4\pi} \oint_S \frac{\vec{M}(\vec{r}') \times d\vec{s}'}{|\vec{r} - \vec{r}'|} \quad \begin{array}{l} \text{variation on} \\ \text{divergence} \\ \text{theorem} \\ \text{(see problem set #6)} \\ \text{or problem 3(a)} \end{array}$$

recall: $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}') d^3 r'}{|\vec{r} - \vec{r}'|}$

note: $\vec{\nabla} \cdot \vec{J}_b = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{M}) = 0$

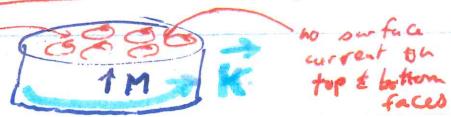
Magnetization bound volume current:

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \vec{J}_M$$

Magnetization bound surface current:

$$\vec{K}_b = \vec{M} \times \hat{n} = \vec{K}_M$$

bound currents cancel



Note: the surface is where the action is (as in dielectrics).

The Auxiliary Field \vec{H}

total current density: $\vec{J} = \vec{J}_{\text{total}} = \vec{J}_b + \vec{J}_{\text{free}}$

$\vec{J}_f \leftarrow$ applied by experimentalist

Ampère's law: $\frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) = \vec{J} = \vec{J}_f + \vec{J}_b = \vec{J}_f + \vec{\nabla} \times \vec{M}$

$$\Rightarrow \vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M} \right) = \vec{J}_{\text{free}}$$

$\underbrace{\quad}_{\vec{H}}$

auxiliary field \vec{H} : $\boxed{\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}}$ obeys Ampère's law

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J}_{\text{free}}}$$



$$\vec{\nabla} \cdot \vec{H} = \frac{1}{\mu_0} \vec{\nabla} \cdot \vec{B} - \vec{\nabla} \cdot \vec{M}$$

↳ $\oint \vec{H} \cdot d\vec{l} = I_{\text{free, enclosed}}$

$$\Rightarrow \vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M} \neq 0$$

especially on surface
where $\vec{K}_b \neq 0$

Linear Magnetic Media

$$\vec{M} = \chi_m \vec{H} \quad \text{where } \chi_m = \text{magnetic susceptibility}$$

\vec{H} not \vec{B}

χ_m is generally very small $\chi_{m_{\text{Cu}}} \approx -10^{-5}, \chi_{m_{\text{Al}}} \approx 2 \times 10^{-5}$

$\chi_m \approx -40 \times 10^{-5}$ exception $\chi_{m_{\text{Fe}}} = 2 \times 10^5$ exception $\chi_{m_{\text{Gadolinium}}} = 0.48$ exception
Pyrolytic Carbon

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) \Rightarrow \boxed{\vec{B} = \underbrace{\mu_0(1+\chi_m)}_{\mu} \vec{H}}$$

$\mu = \mu_0(1+\chi_m)$ = magnetic permeability of material

$$\Rightarrow \boxed{\vec{B} = \mu \vec{H}}$$

⚠ $\vec{\nabla} \cdot \vec{H}_{\text{linear}} = \vec{\nabla} \cdot \left(\frac{1}{\mu} \vec{B} \right) = \frac{1}{\mu} \vec{\nabla} \cdot \vec{B} + \vec{B} \cdot \vec{\nabla} \frac{1}{\mu}$

\Rightarrow inside a uniform material
on a surface of material

$$\Rightarrow \vec{\nabla} \cdot \vec{H}_{\text{linear}} \neq 0 \quad \text{on surfaces}$$

N.B. for most materials (non-magnetic). $\mu \approx \mu_0$

However $\frac{\mu}{\mu_0} \sim 10^5$ or higher

$\frac{\mu}{\mu_0}$ parallel

$\frac{\mu}{\mu_0}$ Ni-Fe-Co $\sim 10^2 - 10^5$

(self) Inductance L

$$\text{magnetic flux} = \phi = \int_S \vec{B} \cdot d\vec{s}$$

$$\text{def: } \phi = L I$$

Note: any current ^{circuit} that produces a B-field has an inductance.

Magnetic Energy

$$E_B = \frac{1}{2\mu_0} \int_V |\vec{B}|^2 d^3r$$

B-Energy stored in an inductor L

$$E_L = \frac{1}{2} L I^2$$

$$\text{note: } E_L = \frac{1}{2} L I^2 = \frac{1}{2\mu_0} \int_V |\vec{B}|^2 d^3r = E_B$$