PHYS 610: Electricity & Magnetism I
Due date: Thursday, February 22, 2018

Problem set #5

1. Maxwell's equations in part – alternate form
Show that the Maxwell's equations embodied in $\partial_\nu G^{\mu\nu} = 0$ can be expressed in terms of the electromagnetic field tensor $F^{\mu\nu}$ in the following manner:

$$\partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} = 0$$

2. Stress-Energy 4-tensor
a) Calculate the individual components of the stress-energy 4-tensor $\Theta^{\mu\nu}$, and show that it can be written as

$$\Theta^{\mu\nu} = \begin{bmatrix} u_{EM} & c\vec{p}_{EM} \\ c\vec{p}_{EM} & \overleftrightarrow{T}_{Maxwell} \end{bmatrix}$$

b) Calculate the invariant $\Theta^{\mu\nu}\Theta_{\mu\nu}$.

c) Consider a collection of charge and current densities interacting with each other electromagnetically. Starting from the general conservation of energy-momentum relation $\partial_\mu \Theta^{\mu\nu} = J_\alpha F^{\alpha\nu}$, derive the following 3-vector relations for energy and momentum conservation:

$$\frac{\partial u_{EM}}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}$$
$$\frac{\partial \vec{p}_{EM}}{\partial t} - \vec{\nabla} \cdot \overleftrightarrow{T}_{Maxwell} = -\vec{f}$$

Here $\vec{f}$ is the electromagnetic force density (i.e. force per unit volume). What does $\vec{J} \cdot \vec{E}$ represent physically? (note: if you need refresher on electromagnetic energy and momentum conservation, then you are encouraged to consult Griffiths chapter 8).

3. Schrodinger equation
Consider the following non-relativistic Lagrangian density:

$$\mathcal{L} = -\frac{\hbar^2}{2m}\left(\vec{\nabla}\psi^*\right) \cdot \left(\vec{\nabla}\psi\right) - V\psi^*\psi + \frac{i\hbar}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi)$$

Where $\psi(\vec{r},t)$ and $\psi^*(\vec{r},t)$ are complex scalar functions that are considered to be independent functions, though they are complex conjugates of eachother. Also, we employ the notation $\dot{\psi} = \frac{\partial}{\partial t}\psi$. Finally, we assume that ψ and ψ^* are null at the boundaries of the system.
Note: You do not need to assume any quantum mechanics to do this problem.

a) Show that the least action principle leads to the Schrodinger equation.

b) Show that the continuous transformation $\psi \rightarrow \psi' = e^{i\alpha}\psi$ and $\psi^* \rightarrow \psi^{*\prime} = e^{-i\alpha}\psi^*$, where α is a small (differential) real constant, is a continuous symmetry of the system.

c) Use the Noether theorem to calculate the Noether current for the system and its associated continuity equation. What is the conserved "charge" of the system? Make the connection with quantum mechanics.

4. Static electromagnetic momentum

Consider a stationary magnetic dipole $\vec{m}$ (ideal) in a static electric field $\vec{E}$.

a) Show that the momentum of the electromagnetic fields of the system is given by

$$\vec{p}_{em} = -\varepsilon_0\mu_0(\vec{m}\times\vec{E})$$

b) You will now calculate the mechanical momentum stored within the magnetic dipole $\vec{m}$ itself (due to the presence of the external electric field $\vec{E}$).

Consider a magnetic dipole that $\vec{m}$ that consists of a current I running in a small square loop with side length a – this dipole is small enough that the electric field does not vary over it. For simplicity (but not loss of generality), choose the orientation (i.e. in-plane rotation angle) of the current loop such that one of its sides is parallel with the component of $\vec{E}$ that lies in the plane of the current loop. You can assume that the current running through the loop consists of positive charges, and that the current running through each segment of the loop is the same (otherwise you get charge buildup). The electric field accelerates and decelerates the charges, depending on the current segment, and thus modifies the relativistic momentum of the charges.

Show that the total mechanical momentum of the charges in the current loop of the magnetic dipole $\vec{m}$ is given by

$$\vec{p}_{em} = +\varepsilon_0\mu_0(\vec{m}\times\vec{E})$$

What is the total momentum of the magnetic dipole in the electric field?

5. Energy versus momentum

Consider the energy U_{EM} and momentum $\vec{P}_{EM}$ stored in the electric and magnetic fields within a volume V. Show that $U_{EM} \geq c|\vec{P}_{EM}|$. In what sort of situation(s) are the two quantities equal.