

Thursday, February 15, 2018

## Electromagnetic Stress-Energy Tensor

[Tangwill 24.4.3]

$$T^{\mu}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A^{\alpha})} \partial_{\nu} A^{\alpha} - \mathcal{L} \delta^{\mu}_{\nu}$$

sum over components

Noether theorem:  $\int T^{\alpha}_{\nu} d^3x$  = invariant and  $\partial_{\mu} T^{\mu}_{\nu} = 0$

Let's calculate  $T^{\mu}_{\nu}$  for  $\mathcal{L}_{EM} = -\frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta}$

$$\frac{\partial \mathcal{L}_{EM}}{\partial(\partial_{\mu} A^{\alpha})} = -\frac{1}{4\mu_0} \frac{\partial}{\partial(\partial_{\mu} A^{\alpha})} \left\{ [\partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha}] [\partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha}] \right\}$$

$$= -\frac{1}{4\mu_0} \frac{\partial}{\partial(\partial_{\mu} A^{\alpha})} \left\{ [g_{\alpha\beta} \partial_{\beta} A^{\alpha} - g_{\beta\alpha} \partial_{\alpha} A^{\beta}] \right.$$

$$\left. [g^{\alpha\sigma} \partial_{\lambda} A^{\sigma} - g^{\sigma\lambda} \partial_{\lambda} A^{\sigma}] \right\}$$

$$= -\frac{1}{4\mu_0} \left\{ \underbrace{g_{12} \delta_{\alpha}^{\mu} \delta_{\alpha}^{\lambda} F^{\sigma\gamma}}_{F^{\mu}_{\alpha}} - \underbrace{g_{\beta\sigma} \delta_{\gamma}^{\mu} \delta_{\alpha}^{\lambda} F^{\sigma\gamma}}_{F_{\alpha}^{\mu}} + \underbrace{F_{\alpha\gamma}^{\mu} g^{\alpha\sigma} \delta_{\lambda}^{\lambda} \delta_{\alpha}^{\sigma}}_{F_{\alpha}^{\mu}} \right. \\ \left. - F_{\alpha\gamma}^{\mu} \underbrace{g^{\beta\sigma} \delta_{\gamma}^{\lambda} \delta_{\alpha}^{\sigma}}_{F^{\mu}_{\alpha}} \right\}$$

$$F^{\mu}_{\alpha} = -F_{\alpha}^{\mu}$$

$$= -\frac{1}{4\mu_0} \left\{ F_{\alpha}^{\mu} + F_{\alpha}^{\mu} + F_{\alpha}^{\mu} + F_{\alpha}^{\mu} \right\}$$

$$= -\frac{1}{\mu_0} F_{\alpha}^{\mu}$$

$$\Rightarrow \left( \frac{\partial \mathcal{L}}{\partial (\partial_\nu A^\alpha)} \right) (\partial_\nu A^\alpha) = -\frac{1}{\mu_0} F^\alpha_\nu \partial_\nu A^\alpha = -\frac{1}{\mu_0} F^{\mu\alpha} \partial_\nu A_\mu$$

According to Noether:

$$\text{Thus } \partial_\mu T^\mu_\nu = 0$$

[objective: "clean up"  $T^\mu_\nu$ ]

$$\Leftrightarrow \partial_\mu \left[ -\frac{1}{\mu_0} F^{\mu\alpha} \partial_\nu A_\alpha - \mathcal{L} \delta^\mu_\nu \right] = 0$$

$$= F_{\nu\alpha} + \partial_\alpha A_\nu \quad \rightarrow F_{\nu\alpha} = \partial_\nu A_\alpha - \partial_\alpha A_\nu$$

$$\Rightarrow \partial_\mu \left[ -\frac{1}{\mu_0} (F^{\mu\alpha} F_{\nu\alpha} + F^{\mu\alpha} \partial_\alpha A_\nu) - \mathcal{L} \delta^\mu_\nu \right] = 0$$

$$\Leftrightarrow -\frac{1}{\mu_0} \left[ \partial_\mu (F^{\mu\alpha} F_{\nu\alpha}) + \underbrace{(\partial_\mu F^{\mu\alpha})(\partial_\alpha A_\nu)}_{=0} + \underbrace{F^{\mu\alpha} \partial_\mu \partial_\alpha A_\nu}_{\substack{\text{antisymmetric} \\ \text{symmetric}}} \right] - \partial_\mu \mathcal{L} \delta^\mu_\nu = 0$$

for Maxwell's equations  
with  $\mathbf{J}^\alpha = 0$   
(no sources)

$$\Rightarrow \partial_\mu \left[ +\frac{1}{\mu_0} F^{\mu\alpha} F_{\nu\alpha} - \frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\gamma} \delta^\mu_\nu \right] = 0$$

definition: Symmetric EM Stress-Energy tensor

$$(\mathbb{H})^\mu_\nu = \frac{1}{\mu_0} F^{\mu\alpha} F_{\nu\alpha} - \frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\gamma} \delta^\mu_\nu$$

note:  $g^{\nu\beta} (\mathbb{H})^\mu_\beta = (\mathbb{H})^{\mu\nu}$  with conservation law  $\partial_\mu (\mathbb{H})^\mu_\nu = 0$

more conveniently,

$$(\mathbb{H})^\mu_\nu = \frac{1}{\mu_0} g^{\nu\beta} F^{\mu\alpha} F_{\beta\alpha} - \frac{1}{4\mu_0} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\gamma}$$

Note:  
 $g^{\nu\beta} \delta^\mu_\beta$   
 $= \delta^{\mu\nu}$

with  $\partial_\mu (\mathbb{H})^\mu_\nu = 0$

for no sources

Note:  $(H)^{\mu\nu} = (H)^{\nu\mu}$  ← symmetric

One can show that

line  $\downarrow$  column  $\downarrow$

$$(H)^{\mu\nu} = \begin{bmatrix} 1 \times 1 & \text{MEM} & | & \vec{c} \vec{P}_{EM} & 1 \times 3 \\ \hline 3 \times 1 & | & | & | & | \\ & \vec{c} \vec{P}_{EM} & | & -\vec{T}_{Maxwell} & \\ & | & | & | & | \\ & & & & 3 \times 3 \end{bmatrix}$$

where  $\left\{ \begin{array}{l} \text{MEM} = (\epsilon_0/2) \vec{E}^2 + (\frac{1}{2\mu_0}) \vec{B}^2 \\ \vec{P}_{EM} = \epsilon_0 (\vec{E} \times \vec{B}) = \frac{\vec{S}_{\text{poynting}}}{c^2} \end{array} \right. = \text{EM energy density}$   
 $\vec{T}_{Maxwell} = \text{EM linear momentum density}$

Sometimes called  $\vec{g}_{EM}$

$$\leftrightarrow \vec{T}_{Maxwell} = -\text{MEM} \delta_{ij} + \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j$$

$$= \epsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} \vec{E}^2 \right) + \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} \vec{B}^2 \right)$$

= Maxwell Stress Tensor

What does  $\partial_\mu (H)^{\mu\nu} = 0$  tell us?

$$\frac{\partial}{\partial x^\mu} (H)^{\mu\nu} = 0$$

$$\left( \underbrace{\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}}_{\left( \frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right)} \right) \left[ \begin{array}{c|c} \text{MEM} & \vec{c} \vec{P}_{EM} \\ \hline \vec{c} \vec{P}_{EM} & -\vec{T}_{Maxwell} \end{array} \right]$$

Equation 1:

$$\frac{1}{c} \frac{\partial}{\partial t} \text{MEM} + \vec{\nabla} \cdot \vec{c} \vec{P}_{EM} = 0$$

Equation 2:

$$\frac{1}{c} \frac{\partial}{\partial t} \vec{c} \vec{P}_{EM} - \vec{\nabla} \cdot \vec{T}_{Maxwell} = 0$$

[ further reading: Griffiths chpt. 8 ]

Equation 1:

$$\frac{\partial \mu_{EM}}{\partial t} + \vec{\nabla} \cdot (\underbrace{c^2 \vec{P}_{EM}}_{\vec{S} = \text{poynting vector}}) = 0$$

$\vec{S}$  = poynting vector

= Energy per unit time per unit area

= Power per unit area

= "intensity", but no EM waves necessary

$$\Leftrightarrow \frac{\partial \mu_{EM}}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0$$

EM energy continuity equation

↳ Local conservation of EM energy

Integrate over a volume  $V$  (finite)

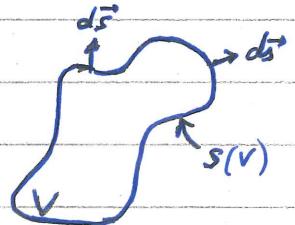
$$\Rightarrow \frac{\partial}{\partial t} \int_V \mu_{EM} d^3r + \int_V \vec{\nabla} \cdot \vec{S} d^3r = 0$$

Some finite  $\rightarrow V$

(1)

$$\Leftrightarrow \frac{\partial}{\partial t} \int_V \mu_{EM} d^3r = - \oint_S \vec{S} \cdot d\vec{s}$$

$S = \text{boundary surface of } V$



$d\vec{s}$  points out

$$\left[ \begin{array}{l} \text{the increase(decrease) in} \\ \text{EM energy in a volume } V \end{array} \right] = \left[ \begin{array}{l} \text{energy flowing in (out)} \\ \text{of the surface of } V \end{array} \right]$$

Equation 2:

$$\frac{\partial}{\partial t} \vec{P}_{EM} - \vec{\nabla} \cdot \vec{T}_{\text{maxwell}} = 0$$

EM momentum continuity equation

↳ Local conservation of momentum

change in  
momentum (in time) density  
= force density

$$\Rightarrow \frac{\partial}{\partial t} \int_V \vec{P}_{EM} d^3r = \int_V (\vec{\nabla} \cdot \vec{T}) d^3r$$

$$\Rightarrow \frac{\partial}{\partial t} \int_V \vec{p}_{EM} d^3r = \oint_{S(V)} \vec{T} \cdot d\vec{s}$$

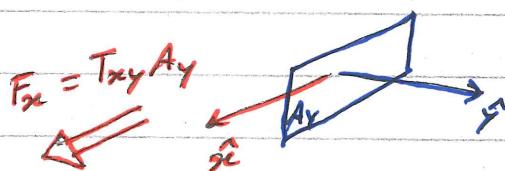
the increase (decrease) in  $\boxed{[EM \text{ momentum in a volume } V]}$  =  $\boxed{[ \text{Force applied to Surface } S(V) \text{ from without (within) } V ]}$

EM force applied to (by)  
a volume  $V$

what is  $\vec{T}$ ?

$\vec{T}$  has dimensions (units) of energy density  $[J/m^3]$   
or (more usefully) Force per unit area  $[N/m^2]$   
↳ i.e. it's an EM "pressure".

$T_{xx}, T_{yy}, T_{zz}$  represent pressure in the  $x, y, z$  directions (like traditional pressure), while  $T_{xy}$  is a shear (i.e. force per unit area in the  $x$  direction acting on a surface with direction  $y$ ).



Let's add in  $\vec{t}_{\mu\nu}$  charges and currents

$\partial_\mu (\mathbb{H})^{\mu\nu} = 0$  is valid for regions of space-time where  
 $\epsilon = 0, \vec{J} = 0$   
 $\hookrightarrow \vec{J}^\mu = 0$

Maxwell's equations with sources :  $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$

Q:  $\partial_\mu \mathcal{H}^{\mu\nu} = ?$  with charges & currents #6

thus  $\mu_0 \partial_\mu (\mathcal{H})^{\mu\nu} = \mu_0 \partial_\mu \left[ \frac{1}{\mu_0} \partial^\nu F^{\mu\alpha} F_{\beta\alpha} - \frac{1}{4\mu_0} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right]$

$$= g^{\nu\beta} \underbrace{\left( \partial_\mu F^{\mu\alpha} \right)}_{-\mu_0 J^\alpha} F_{\beta\alpha} + g^{\nu\beta} F^{\mu\alpha} \left( \partial_\mu F_{\beta\alpha} \right) - \frac{1}{4} g^{\mu\nu} \left( \partial_\mu F_{\alpha\beta} \right) F^{\alpha\beta} - \frac{1}{4} g^{\mu\nu} \left( \partial_\mu F^{\alpha\beta} \right) F_{\alpha\beta}$$

from Maxwell's equations

$$= -\mu_0 \underbrace{J^\alpha F^{\nu\alpha}}_{J_\alpha F^{\nu\alpha}} + \underbrace{F^{\mu\alpha} \left( \partial_\mu F^{\nu\alpha} \right)}_{E_{\mu\alpha} (\partial^\mu F^{\nu\alpha})} - \frac{1}{4} \underbrace{\left( \partial^\nu F_{\alpha\beta} \right) F^{\alpha\beta}}_{(\partial^\nu F^{\alpha\beta}) F_{\alpha\beta}} - \frac{1}{4} F_{\alpha\beta} \left( \partial^\nu F^{\alpha\beta} \right)$$

stopped here

$$= -\mu_0 F^{\nu\alpha} J_\alpha + \underbrace{E_{\mu\alpha} (\partial^\mu F^{\nu\alpha})}_{\text{rename } \mu \rightarrow \sigma \atop \alpha \rightarrow \gamma} - \frac{1}{2} F_{\alpha\beta} \left( \partial^\nu F^{\alpha\beta} \right)$$

$$= -\mu_0 F^{\nu\alpha} J_\alpha + \frac{1}{2} \left[ F_{\alpha\gamma} \left( \partial^\sigma F^{\nu\gamma} \right) + \underbrace{F_{\alpha\gamma} \left( \partial^\sigma F^{\nu\gamma} \right)}_{-F^{\sigma\nu}} - F_{\alpha\gamma} \left( \partial^\nu F^{\sigma\gamma} \right) \right]$$

$$\text{rename } \sigma \rightarrow \eta \atop \eta \rightarrow \sigma$$

$$F_{\alpha\eta} \left( \partial^\eta F^{\nu\sigma} \right) - F_{\alpha\eta} \left( \partial^\eta F^{\nu\sigma} \right)$$

$$= -\mu_0 F^{\nu\alpha} J_\alpha + \frac{1}{2} F_{\alpha\eta} \left\{ \underbrace{\partial^\sigma F^{\nu\eta} + \partial^\eta F^{\nu\sigma} + \partial^\nu F^{\sigma\eta}}_{=0} \right\}$$

equivalent to  $\partial_\mu G^{\mu\nu} = 0$   
(see Problem Set #5)

$$\Rightarrow \boxed{\partial_\mu (\mathcal{H})^{\mu\nu} = - F^{\nu\alpha} J_\alpha = J_\alpha F^{\nu\alpha}}$$

Conservation Mechanical+EM  
Energy-momentum