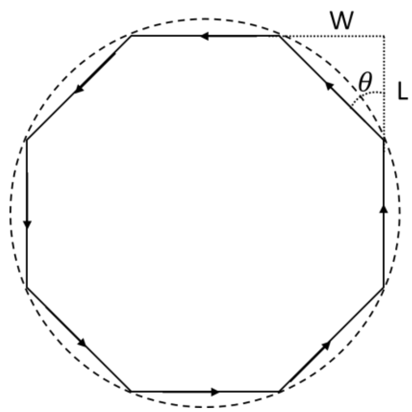
PHYS 610: Electricity & Magnetism I
Due date: Thursday, March 15, 2018

Problem set #6

1. Thomas Precession: a more physical derivation

In this problem, you will calculate the Thomas precession frequency (in the lab frame) for an electron in circular motion with a method due to Smoot and Purcell. We begin approximating circular motion with an *N*-sided polygon, as shown in the figure below (W, L, and θ are all in the lab frame).



a) Consider the electron as it travels along one of the straight segments. Calculate, in the frame of the electron, the angle θ' by which the electron will rotate in the next segment in terms of W' and L' (in the frame of the electron) for *N* large. You can then write θ' in terms of W and L in the lab frame. Next, give the relationship between θ' and θ. The electron has an orbital velocity of *v* and an orbital frequency ω_{orbit} in the lab frame.

b) We define $\Delta\theta = \theta'_{total} - \theta_{total}$ as the difference in accumulated rotation phase over the course of one orbit between the electron's frame and the lab frame. Here $\theta_{total} = 2\pi$ is the rotation phase accumulated in the lab frame over the course of one orbit (in the lab frame). Calculate $\Delta\theta$ in terms of the relativistic factor γ.

c) Write down the relationship between the orbital periods in the lab frame and the electron's frame. We define the Thomas precession frequency ω_{Thomas} (in the lab frame) as the difference in accumulated rotation phase per orbit time: $\omega_{Thomas} = \Delta\theta/T$. Show that $\omega_{Thomas} = \omega_{orbit}(\gamma - 1)$.

d) Show that for circular motion $\omega_{Thomas} = av/(2c^2)$, where a is the centripetal acceleration of the electron in the lab frame.

2. Magic Gamma

Consider the Thomas-BMT equation for the longitudinal spin polarization of a particle subject to a magnetic field $\vec{B}$ and electric field $\vec{E}$ (in the lab frame):

$$\frac{ds_{\parallel}}{dt} = -\frac{q_e}{m}\vec{s}_{\perp} \cdot \left[a\hat{\beta} \times \vec{B} + \left(a - \frac{1}{\gamma^2 - 1}\right)\vec{E}\right]$$

Where $a = (g-2)/2$ is the anomalous magnetic moment, g is the g-factor of the particle, (relating its magnetic moment and spin), and $\vec{\beta} = \vec{v}/c$ relates the velocity $\vec{v}$ of the particle. The longitudinal and transverse spin polarization components are given by $s_{\parallel} = \vec{s} \cdot \hat{\beta}$ and $\vec{s}_{\perp} = \vec{s} - s_{\parallel}\hat{\beta}$, respectively.

a) Show that there is a "magic" velocity (and γ) such that the electric field does not influence the longitudinal spin polarization.

b) Calculate the relativistic energy for a muon travelling at this "magic" γ, and compare it with the storage ring energy for the FermiLab experiment which is measuring its g-factor (or "g-2").

3. Variation on the Divergence Theorem

Prove the following two integral theorems:

a) $\int_V \left(\vec{\nabla} \times \vec{F}\right) d^3r = \int_S \vec{\text{F}} \times d\vec{s}$

b) $\int_V \vec{\nabla} f d^3r = \int_S f d\vec{s}$

Where f is a scalar function, $\vec{F}$ is a vector function, and S is the bounding surface for a volume V.

Hint: You may want to consider "multiplying" the appropriate field by a constant vector field.

4. Variation on Stokes' theorem

Prove the following integral theorem: $\int_S \hat{n} \times \vec{\nabla} f \, dS = \int_C f \, \overrightarrow{dl}$

Where f is a scalar function, S is surface with contour C, $\hat{n}$ is a unit vector locally perpendicular to S, and $\overrightarrow{dl}$ is a differential line element along C.

5. Green's identities

a) Use the divergence theorem to prove *Green's first identity*:

$$\int_V \left[\phi \vec{\nabla}^2 \psi + \vec{\nabla}\phi \cdot \vec{\nabla}\psi\right] d^3r = \int_S \phi \vec{\nabla}\psi \cdot \overrightarrow{dS}$$

$\phi(\vec{r})$ and $\psi(\vec{r})$ are arbitrary (well-behaved) scalar functions, and *V* is a volume with surface *S*.

b) Prove *Green's second identity*:

$$\int_V \left[\phi \vec{\nabla}^2 \psi - \psi \vec{\nabla}^2 \phi\right] d^3r = \int_S \left[\phi \vec{\nabla}\psi - \psi \vec{\nabla}\phi\right] \cdot \overrightarrow{dS}$$

6. Mean value theorem for electrostatics

Consider a function $f(\vec{r})$ that obeys Laplace's equation $\vec{\nabla}^2 f = 0$. Show that $f(\vec{r})$ obeys the following average rule: The value of $f(\vec{r})$ at any point $\vec{r}$ is equal to the average of $f(\vec{r})$ over the surface of any sphere centered on $\vec{r}$.

Note: this result shows that $f(\vec{r})$ can have no local maximum or minimum, only saddle points at most.