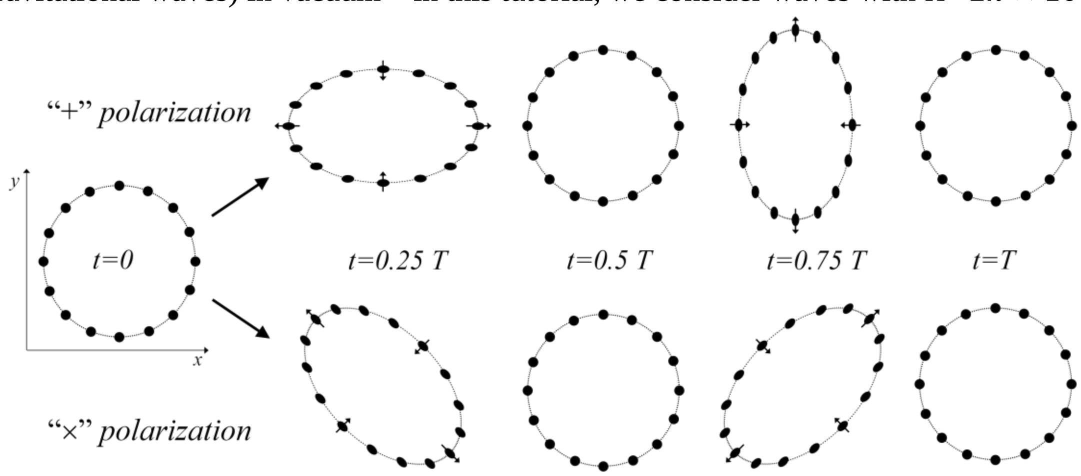
Tutorial problem: Interferometric detection of gravitational waves

This tutorial problem is designed to teach the special relativity and general relativity involved in gravitational wave detection with an optical interferometer, though some special relativity knowledge is assumed. The questions posed are not mathematically challenging, but they make use of some non-intuitive concepts. The tutorial is based on a paper by P. R. Saulson [Am. J. Phys. ***65****, 501 (1997)], as well as discussions with Prof. J. Erlich.*

Gravitational wave review: We consider a weak gravitational wave propagating in the +*z* direction with its polarization axes aligned with *x* and *y* axes. This weak gravitational wave can be described as a travelling perturbation $h_{\mu\nu}$ to the flat-space metric $g_{\mu\nu}$ (in the "traceless transverse" gauge):

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{+} & h_{\times} & 0 \\ 0 & h_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} e^{i(\mathrm{K}z-\Omega t)}$$

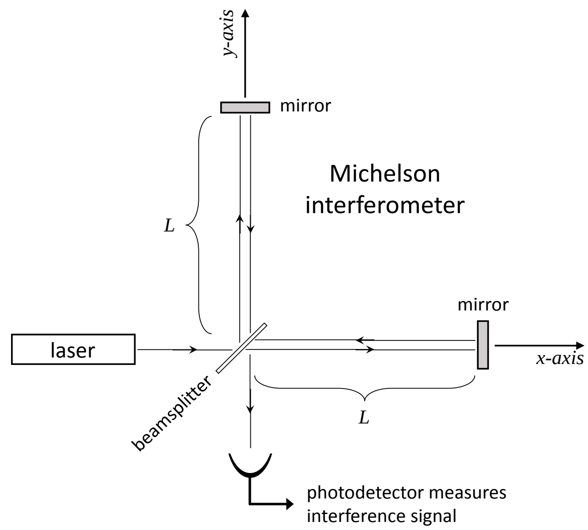
where h_{+} and $h_{\times}$ are the amplitudes of the "plus" polarization (along *x* and *y* axes) and "cross" polarization (along the axes at 45° degrees to the *x* and *y* axes) of the quadrupole deformation induced by the gravitational wave, as shown in the figure below. The h_{+} and $h_{\times}$ amplitudes represent the fractional stretching and compressing (i.e. strain) of distances in the x-y plane, which is typically extremely small – the largest gravitational strain measured by LIGO is $h{\sim}10^{-21}$. The wavevector K and frequency Ω of the wave are related by $\Omega/\mathrm{K} = c$, where *c* is the speed of light (and gravitational waves) in vacuum – in this tutorial, we consider waves with $\Omega{\sim}2\pi \times 100$ Hz.



The quadrupole deformation of space by the gravitational wave can be expressed as a linear combination of the "+" and "×" polarizations (i.e deformations). T is the period of the wave.

In the transverse traceless gauge coordinate system of general relativity, the coordinate tickmarks are indicated by a matrix of free test masses, which are subject to gravity (and gravitational waves). If we define "coordinate distance" by the number of tickmarks between two given free masses, then gravity cannot alter the "coordinate distance" between them. In contrast, the "proper distance" is defined by the metric through the use of the infinitesimal space-time separation $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$, which is altered by gravity and is also reference frame invariant.

Interferometer review: A Michelson interferometer, such as the one used by the LIGO gravitational wave observatory, measures the optical path length difference (i.e. phase difference) between two beams of light that travel down two perpendicular arms of equal length L. The two beams are produced by directing laser light on 50/50 beamsplitter. The light is also recombined when the beams from two arms return to it: depending on the phase difference between the two arms, roughly half the light returns towards the laser, while the remaining light is directed on a photodetector.



Michelson interferometer for detecting gravitational waves. The two mirrors and the beamsplitter are suspended so that they act as free masses.

Importantly, the two mirrors and the beamsplitter are suspended so that they behave as free masses when subject to a gravitational force or wave, i.e. L can be stretched or compressed by a gravitational wave. At LIGO, the length of the arms is L = 4 km, but the addition of Fabry-Perot cavities in each arm (not shown in figure) results in effective arm lengths of L=1120 km. LIGO uses laser light with a wavelength of 1064 nm.

Questions

A. General relativity analysis using the transverse traceless gauge

a1) We consider a "plus" polarized gravitational wave.

- Show that two nearby space-time events with infinitesimal separations in time, *dt*, and space, *dx* and *dy*, have a frame invariant space-time separation *ds* given by: $ds^2 = -c^2dt^2 + (1 + h_+(t))dx^2 + (1 - h_+(t))dy^2$.

Here the time dependence of $h_+(t)$ is given by the propagating plane wave nature of the gravitational wave.

a2) A central tenet of relativity is that the local speed of light in vacuum is the same in all inertial frames (including those under the influence of gravity), so that for a light ray trajectory $ds = 0$. Consider a light ray (or wave) that propagates along the *x*-axis or *y*-axis (in the interferometer's reference frame).

- Show that the time for the light ray to propagate along these axes of the interferometer is given by

$$\tau_x \simeq \frac{1}{c}\int\left(1 + \frac{h_+(t)}{2}\right)dx \text{ and } \tau_y \simeq \frac{1}{c}\int\left(1 - \frac{h_+(t)}{2}\right)dy$$

Important: In the transverse traceless gauge picture, the tickmarks of the coordinate system are indicated by free masses (which are subject to gravity). These tickmark free masses "move" with the gravitational wave, but since there is no change in the number of tickmarks between free masses, the two integrals should be done over the coordinates from x=0 to x=L (and y=0 to y=L).

a3) Typically the period of the gravitational wave $T = 2\pi/\Omega$ is quite long compared to the round-trip travel time for a light ray in an interferometer arm, so that the interferometer arm length is quasi-static over the duration of propagation of a light pulse in either arm.

- If we direct simultaneously two light pulses down each interferometer arm (starting at the beamsplitter), then show that the maximum difference in arrival times for the two light pulse back at the beamsplitter is given by $\Delta\tau = 2Lh_+/c$. Estimate $\Delta\tau$ in seconds and the corresponding spatial shift between the two pulses in meters.

- If instead we direct a continuous wave laser beam with wavelength λ into the beamsplitter and down the two interferometer arms, then show that the maximum phase difference between the two light waves when they recombine at the beamsplitter is given by $\Delta\phi = 4\pi Lh_+/\lambda$. Estimate $\Delta\phi$ in radians.

B. "Rubber Ruler" Paradox: *Since light waves are stretched by gravitational waves, then how can we use light as a ruler to detect gravitational waves?* This section resolves this paradox.

Quasi-Newtonian view of weak gravitational waves

Question: What does it mean that space is stretched (compressed) by a gravitational wave?
Answer: The distance between two nearby free-floating test masses will increase (decrease) by a factor of $1 + h_+/2$, as measured by a rigid ruler. In the quasi-Newtonian picture, the free-floating tests masses move due to the gravitational "force" from the gravitational wave. The rigid ruler does not change length because the electromagnetic interactions holding its atoms together completely overwhelm the gravitational force. Since a light wave is subject to gravity, it behaves as a train of free masses, and so the wavelength of a light wave is increased (decreased) also by a factor of $1 + h_+/2$ by a passing gravitational wave.

b1) Naïve calculation
We examine the interferometer when the x-axis is maximally stretched and the y-axis is maximally compressed by the gravitational wave (quasi-static picture).

- Calculate the length of each axis.

- Calculate the wavelengths of light waves that are already propagating within each arm within the interferometer, λ_{x-arm} and λ_{y-arm}.

- Calculate the length of each arm in units of its wavelength (i.e. how many optical spatial cycles fit into each arm).

- Show that the difference in spatial optical cycles and phase between the two arms is zero.

b2) A more complete calculation
The calculations in b1) are not wrong, but they suggests that the gravitational wave does not generate a phase difference between light in the two interferometer arms. This conclusion is false, because the calculation is not complete.

- The laser used to feed the interferometer is based on a cavity with a rigid length. The length of the cavity is an integer multiple of half of the laser's wavelength, i.e. $L_{laser\ cavity} = n\lambda_{laser}/2$, where n is an integer. Is the wavelength λ_{laser} of the light just as it emerges from the laser longer, shorter, or unchanged during the passage of the gravitational wave (at maximum stretch)?

- After waiting for light from the laser to travel from the beamsplitter to the mirrors and back (i.e. after waiting for the interferometer to be filled with new light), calculate the length of each interferometer arm in units of λ_{laser} (at maximum stretch for the *x*-arm and maximum compression for the *y*-arm).

- Calculate the difference in optical cycles between the two arms and show that it corresponds to a phase difference of $\Delta\phi = 4\pi L h_+/\lambda_{laser}$.