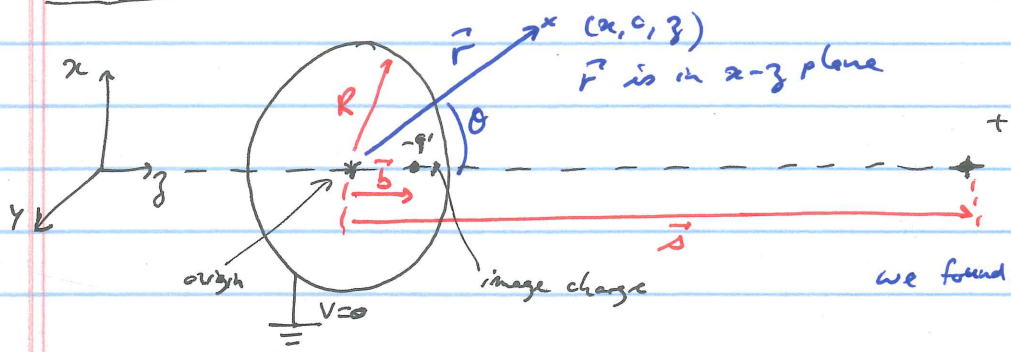
Thursday, March 22, 2018

Method of Images: point charge outside a conducting sphere



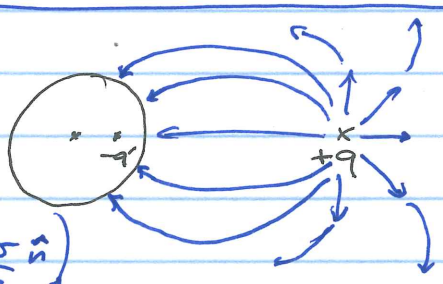
we found: $-q' = -qR/s$

$b = R^2/s$

Last time we showed that

$$V_{total}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{r^2+s^2-2sr\cos\theta}} - \frac{R/s}{\sqrt{r^2+\frac{R^4}{s^2}-2\frac{R^2}{s}r\cos\theta}}\right\}$$

Surface charge density: σ



$$\left.\frac{\partial V_{total}}{\partial r}\right|_{r=R} = -\frac{\sigma}{\varepsilon_0} \qquad \left(\text{i.e. } \vec{E}_{sphere} = \frac{\sigma}{\varepsilon_0}\hat{n}\right)$$

$$\Rightarrow \left.\frac{-\partial V}{\partial r}\right|_{r=R} = \frac{-q}{4\pi\varepsilon_0}\left\{\frac{(-\frac{1}{2})(2r-2s\cos\theta)}{[r^2+s^2-2sr\cos\theta]^{3/2}} - \frac{(-\frac{1}{2})(R/s)\left(2r-2\frac{R^2}{s}\cos\theta\right)}{\left[r^2+\frac{R^4}{s^2}-2\frac{R^2}{s}r\cos\theta\right]^{3/2}}\right\}_{r=R}$$

$$= \frac{-q}{4\pi\varepsilon_0}\left\{-\frac{(R-s\cos\theta)}{[R^2+s^2-2sR\cos\theta]^{3/2}} + \frac{\left(R^2/s - R^3/s^2\cos\theta\right)}{\left[R^2+R^4/s^2-2R^3/s\cos\theta\right]^{3/2}}\right\}$$

$$= \frac{-q}{4\pi\varepsilon_0}\left\{-\frac{(R-s\cos\theta)}{s^3\left[\frac{R^2}{s^2}+1-2\frac{R}{s}\cos\theta\right]^{3/2}} + \frac{\left(R^2/s - R^3/s^2\cos\theta\right)}{R^3\left[1+R^2/s^2-2R/s\cos\theta\right]^{3/2}}\right\}$$

skip

$$= \frac{-q}{4\pi\varepsilon_0}\frac{1}{R^3s^3\left[1+\frac{R^2}{s^2}-2\frac{R}{s}\cos\theta\right]^{3/2}}\left\{-\left(R^4-sR^3\cos\theta\right)+\left(R^2s^2-R^3s\cos\theta\right)\right\}$$

$R^2s^2-R^4$

$$\sigma = \varepsilon \left.\frac{\partial V}{\partial r}\right|_{r=R} = \frac{-q}{4\pi} \frac{1}{\left[1+\frac{R^2}{s^2}-2\frac{R}{s}\cos\theta\right]^{3/2}} \qquad \left\{\frac{R^2 s^2 - R^4}{R^3 s^3}\right\}$$

skip

$$\frac{1}{Rs} - \frac{R}{s^3}$$

$$\frac{1}{Rs}\left(1-\frac{R^2}{s^2}\right)$$

$$\frac{1}{R^2}\,\frac{R}{s}\left(1-\frac{R^2}{s^2}\right)$$

$$\Rightarrow \quad \boxed{\sigma = \frac{-q}{4\pi R^2} \frac{R/s\left(1-\frac{R^2}{s^2}\right)}{\left[1+\frac{R^2}{s^2}-2\frac{R}{s}\cos\theta\right]^{3/2}}}$$

note: $Q_{sphere} = \int_0^{2\pi}\int_0^{\pi} \sigma(\theta)\, R^2 \sin\theta\, d\theta\, d\phi$

$$= 2\pi R^2 \left(\frac{-q}{4\pi R^2}\right)\left(R/s\right)\left(1-R^2/s^2\right) \int_0^{\pi} \frac{\sin\theta\, d\theta}{\left[1+\frac{R^2}{s^2}-\frac{2R}{s}\cos\theta\right]^{3/2}}$$

$$\frac{1}{\left(+2\frac{R}{s}\right)}\,(-2)\,\frac{1}{\left[1+\frac{R^2}{s^2}-2\frac{R}{s}\cos\theta\right]^{1/2}}\Bigg|_0^{\pi}$$

skip

$$= -\frac{s}{R}\left[\frac{1}{\left(1+\frac{R^2}{s^2}+2\frac{R}{s}\right)^{1/2}} - \frac{1}{\left(1+\frac{R^2}{s^2}-2\frac{R}{s}\right)^{1/2}}\right]$$

$$= -\frac{s}{R}\left[\frac{1}{1+\frac{R}{s}} - \frac{1}{1-\frac{R}{s}}\right]$$

$$= -\frac{s}{R}\left[\frac{\left(1-R/s\right)-\left(1+R/s\right)}{1-R^2/s^2}\right] = \frac{+2}{1-R^2/s^2}$$

$$Q_{sphere} = \cancel{2\pi R^2}\left(\frac{-q}{\cancel{4\pi R^2}}\right)\left(\frac{R}{s}\right)\cancel{\left(1-\frac{R^2}{s^2}\right)}\left(\frac{+\cancel{2}}{\cancel{1-\frac{R^2}{s^2}}}\right)$$

$$= -q\frac{R}{s}$$

$$\Rightarrow \boxed{Q_{sphere} = -q\frac{R}{s} = -q'}$$ ✓

The sphere and point charge attract each other with force:

$$\boxed{F = \frac{1}{4\pi\varepsilon_0}\frac{q^2\frac{R}{s}}{\left(s-\frac{R^2}{s}\right)^2}}$$

Green's Function Method [Jackson 1.10, 2.6; Zangwill 8.4, 8.5]

General approach

Consider a differential equation of the form

$$D[u(\vec{r})] = f(\vec{r})$$

- D: Linear differential operator (typically, highest derivative is 2nd order, e.g. $\vec{\nabla}^2$ ← Poisson's eq.)
- $u(\vec{r})$: unknown
- $f(\vec{r})$: source term (known)

We want to solve for $u(\vec{r})$ such that it also satisfies a boundary condition on some surface S.
(e.g. $u(\vec{r} \in S) = 0$)

The Green's function for this problem is the function $G(\vec{r}, \vec{r}')$ such that

$$D[G(\vec{r}, \vec{r}')] = N\,\delta(\vec{r} - \vec{r}')$$

- D acts on "$\vec{r}$"
- N: normalization constant ($N = 1$ for now)

$$\Rightarrow \quad D[G(\vec{r}, \vec{r}')] \cdot f(\vec{r}') = \delta(\vec{r} - \vec{r}') \cdot f(\vec{r}')$$

$$\Rightarrow \quad \int_V D[G(\vec{r}, \vec{r}')] \cdot f(\vec{r}')\, d^3r' = \int_V \delta(\vec{r} - \vec{r}') \cdot f(\vec{r}')\, d^3r'$$

$D[\,]$ act only on $\vec{r}$ and is linear

$$\Rightarrow \quad D\left[\int_V G(\vec{r}, \vec{r}') \cdot f(\vec{r}')\, d^3r'\right] = f(\vec{r})$$

a solution to the original differential equation is thus

$$u(\vec{r}) = \int_V G(\vec{r},\vec{r}') \cdot f(\vec{r}')\, d^3r'$$

Advantage: Once $G(\vec{r},\vec{r}')$ is determined then the solution $u_f(\vec{r})$ for any source term $f(\vec{r})$ is straightforward by computing the above convolution integral. analytically or numerically

Disadvantage: Determining $G(\vec{r},\vec{r}')$ can be difficult.

Reciprocity Theorem: $G(\vec{r},\vec{r}') = G(\vec{r}',\vec{r})$

Simple example: Consider Poisson's equation in free space (i.e. boundary condition is $V((r\to\infty)\to 0)$)

$$\underbrace{\vec{\nabla}^2 V(\vec{r})}_{u(\vec{r})} = \underbrace{-\frac{\rho(\vec{r})}{\varepsilon_0}}_{f(\vec{r})}$$

We know that (see March 1 lecture)

$$\vec{\nabla}^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}')$$

thus $G(\vec{r},\vec{r}') = -\frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}'|}$ with $\nabla^2 G(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')$

and so

$$V(\vec{r}) = \int_V G(\vec{r},\vec{r}')\cdot f(\vec{r}')\, d^3r'$$

$$= \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d^3r' \quad \text{(see March 15 lecture)}$$

(Not a surprise! you've been using Green's functions without realizing it!)

Q: the formula for $u(\vec{r})$, i.e. $V(\vec{r})$, is really convenient, why don't we use it more? (instead of separation of variables or method of images)

A: This formula must be modified when there is a more complicated boundary condition.

↳ yields a more complicated Green's function method

Green's functions for Electrostatic systems → $D[\,] = \vec{\nabla}^2$

(i.e. how to include more complicated boundary conditions)

Consider Green's 2nd identity (see problem set #6)

$$\int_V \left[\Phi(\vec{r}')\,\vec{\nabla}^2_{r'}\Psi(\vec{r}') - \Psi(\vec{r}')\,\vec{\nabla}^2_{r'}\Phi(\vec{r}')\right] d^3r' = \int_S \left[\Phi(\vec{r}')\,\vec{\nabla}_{r'}\Psi(\vec{r}') - \Psi(\vec{r}')\,\vec{\nabla}_{r'}\Phi(\vec{r}')\right]\cdot d\vec{s}'$$

$\hat{s} = \hat{n}$ points out of volume

with $\begin{cases} \Phi(\vec{r}') = V(\vec{r}') \\ \Psi(\vec{r}') = G(\vec{r},\vec{r}') \end{cases}$, i.e. $\vec{\nabla}^2_{r'}\, G(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')$

$$\Rightarrow \underbrace{\int_V V(\vec{r}')\,\underbrace{\vec{\nabla}^2_{r'}\, G(\vec{r},\vec{r}')}_{\delta(\vec{r}-\vec{r}')}\, d^3r'}_{V(\vec{r})} - \int_V G(\vec{r},\vec{r}')\,\underbrace{\vec{\nabla}^2_{r'}\, V(\vec{r}')}_{-\frac{\rho(\vec{r}')}{\varepsilon_0}}\, d^3r'$$

$$= \int_S V(\vec{r}')\,\underbrace{\vec{\nabla}_{r'}\, G(\vec{r},\vec{r}')\cdot d\vec{s}'}_{\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\, ds'} - \int_S G(\vec{r},\vec{r}')\,\underbrace{\vec{\nabla}_{r'}\, V(\vec{r}')\cdot d\vec{s}'}_{\frac{\partial V(\vec{r}')}{\partial n'}\, ds'}$$

$$\Rightarrow \boxed{V(\vec{r}) = -\int_V G(\vec{r},\vec{r}')\,\frac{\rho(\vec{r}')}{\varepsilon_0}\, d^3r' \;\underbrace{+\int_S V(\vec{r}')\,\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\, ds' - \int_S G(\vec{r},\vec{r}')\,\frac{\partial V(\vec{r}')}{\partial n'}\, ds'}_{\text{boundary conditions}}}$$

Dirichlet boundary conditions

$$G_D(\vec{r}\in S, \vec{r}'\in V) = 0$$

$$\text{or } G_D(\vec{r}'\in S, \vec{r}\in V) = 0$$

then $$\int_S G(\vec{r},\vec{r}')\,\frac{\partial V(\vec{r}')}{\partial n'}\,da' = 0$$

Neumann b.c.

$$\frac{\partial G(\vec{r},\vec{r}')}{\partial n'} = 0 \quad \text{for } \vec{r},\vec{r}'\in S$$

i.e. $$\int_S V(\vec{r}')\,\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\,da' = 0$$

consequently,

$$V(\vec{r}) = -\int_V G_D(\vec{r},\vec{r}')\,\frac{\rho(\vec{r}')}{\varepsilon_0}\,d^3r' + \int_S V(\vec{r}')\,\frac{\partial G_D(\vec{r},\vec{r}')}{\partial n'}\,da'$$

($V(\vec{r}')$: specifies b.c. on S)

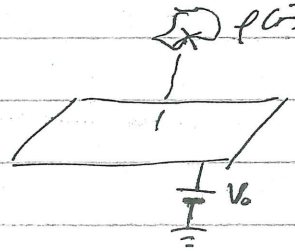
for Dirichlet Green's function

$\hat{n}'$ points out of volume

physical interpretation of $G_D(\vec{r},\vec{r}')$

$G_D(\vec{r},\vec{r}')$ is the electrostatic "potential" at any point $\vec{r}$ in V bounded by a grounded conducting surface due to a point charge ($q=1$) placed at $\vec{r}'$ in V.

Example: Compute $V(\vec{r})$ and $G_D(\vec{r},\vec{r}')$ for an infinite conducting plane at $z=0$ held at potential V_0.



We can get $G_D(\vec{r},\vec{r}')$ by the method of images.

$$\nabla^2 G_D(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')$$

and $$G_D(\vec{r},\vec{r}')\Big|_{z=0} = 0$$

for volume V: $z > 0$

$$G(\vec{r},\vec{r}') = -\frac{1}{4\pi}\left[\frac{1}{|\vec{r}-\vec{r}'|} - \frac{1}{|\vec{r}-\vec{r}'_{image}|}\right]$$

Stopped here

with $\vec{r}'_{image} = (x', y', -z') \notin V$ ($z' > 0$)

$$\Rightarrow G(\vec{r},\vec{r}') = -\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right]$$

the Dirichlet b.c. is satisfied $G_D(\vec{r},\vec{r}')\Big|_{z'=0} = 0$ (or $z'=0$)

for the surface integral, we need to calculate

$$\frac{\partial G_D(\vec{r},\vec{r}')}{\partial n'} = -\frac{\partial G_D}{\partial z'}\Big|_{z'=0}$$

since $\hat{n}'$ points out of V ($z>0$), so $\hat{n}' = -\hat{z}'$ (i.e. points towards negative z')

$$= +\frac{1}{4\pi}\left\{\frac{(\tfrac{1}{2})\, z\,(z-z')(-1)}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}} + \frac{(\tfrac{1}{2})\, z\,(z+z')(+1)}{\left[(x-x')^2+(y-y')^2+(z+z')^2\right]^{3/2}}\right\}_{z'=0}$$

$$= +\frac{1}{4\pi}\frac{2z}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}}$$