PHYS 622: Quantum Mechanics II
Due date: none … do not turn in.

## Practice Problems: Dirac Equation

**1. Dirac matrices, part 1** [Sakurai & Napolitano 8.8]

Prove that the traces of the $\gamma^\mu$, $\alpha_i$ (i=1,2,3), $\beta$ matrices are all zero.

**2. Dirac matrices, part 2** [Sakurai & Napolitano 8.9]

a) Derive the matrices $\gamma^\mu$ from 8.2.10 and show that they satisfy the Clifford algebra 8.2.4

b) Show that

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

$$\gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}$$

Where $i$=1,2,3, $I$ is the 2×2 identity matrix, and $\sigma_i$ are the 2×2 Pauli matrices.

**3. Plane wave solutions of the Dirac equation**

We will look for the general plane wave solutions $\Psi(\vec{r},t) = \vec{u}e^{i(\vec{k}\cdot\vec{r}-\frac{E}{\hbar}t)}$ for a particle of mass $m$, where $\Psi$ is a 4-component Dirac spinor, $\vec{u}$ is a 4-component vector, $E$ is the energy, and $\vec{p} = \hbar\vec{k}$ is the momentum.

a) Show that the Dirac equation can be put into the matrix form

$$\begin{bmatrix} E-mc^2 & 0 & -cp_z & -c(p_x-ip_y) \\ 0 & E-mc^2 & -c(p_x+ip_y) & cp_z \\ -cp_z & -c(p_x-ip_y) & E+mc^2 & 0 \\ -c(p_x+ip_y) & cp_z & 0 & E+mc^2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

b) Show that the four normalized solutions of this equation are

$$\Psi_{+,R}(\vec{r},t) = Ae^{i(\vec{k}\cdot\vec{r}-\frac{E_+}{\hbar}t)} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E_++mc^2} \\ \frac{c(p_x+ip_y)}{E_++mc^2} \end{pmatrix} \qquad \Psi_{+,L}(\vec{r},t) = Ae^{i(\vec{k}\cdot\vec{r}-\frac{E_+}{\hbar}t)} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E_++mc^2} \\ \frac{-cp_z}{E_++mc^2} \end{pmatrix}$$

$$\Psi_{-,R}(\vec{r},t) = Ae^{i\left(\vec{k}\cdot\vec{r}-\frac{E_+}{\hbar}t\right)} \begin{pmatrix} \frac{cp_z}{E_- - mc^2} \\ \frac{c(p_x+ip_y)}{E_- - mc^2} \\ 1 \\ 0 \end{pmatrix} \qquad \Psi_{-,L}(\vec{r},t) = Ae^{i\left(\vec{k}\cdot\vec{r}-\frac{E_+}{\hbar}t\right)} \begin{pmatrix} \frac{c(p_x-ip_y)}{E_- - mc^2} \\ \frac{-cp_z}{E_- - mc^2} \\ 0 \\ 1 \end{pmatrix}$$

Where $E_{\pm} = \pm\sqrt{m^2c^4 + c^2p^2}$ are the associated energies of the Dirac spinors.

Also, $A = \sqrt{\frac{E_+ + mc^2}{2E_+}}$ is a normalization factor.

**4. Lorentz boost of a plane wave**

Consider an electron of mass $m$ at rest in a reference frame R. Write down the Dirac spinors $\Psi_0(\vec{r},t)$ for the electron in the R frame.

Next consider an observer in a frame R’ moving with velocity $v = v_z\hat{z}$ with respect to R. Use the Lorentz boost transformation matrix S for Dirac spinors, $\Psi'(x'^{\mu}) = S\Psi(x^{\mu})$, to calculate $\Psi'_0\left(\overrightarrow{r'},t'\right)$ for the electron in the R’ frame, and verify that it is consistent with the Dirac spinors for a plane wave obtained from the formulas in problem 3b. You may choose to work with a single one of the four possible Dirac spinors.