Wednesday, August 28, 2024

Intro to Error Analysis

I) Measurements

1) Direct measurements

Explicitly measure the quantity of interest.

ex: - Measure length with a ruler.
- Measure temperature with a thermometer.

2) Indirect measurements

Measure quantities related to the quantity of interest and then determine this quantity from the measured quantities (usually by calculation).

Ex: Measure area A of a rectangle.

↳ $A = \text{length} \times \text{width}$

↳ measure length & width → calculate area A.

II) Uncertainty/error of measurements

All measurements do not give a single value for a quantity, but a range of values.

In physics, we quote the central value ("most likely value") and then give the range (i.e. error bar) for other acceptable values for the quantity.

Ex: If you measure a length with a ruler, then you will be uncertain of the exact length to a fraction of the smallest tick marks (tick width).

$L = 17.8 \pm 0.1$ cm $= 17.8(1)$ cm $= 17.8$ c

(if you are really careful: $L = 17.85 \pm 0.05$ cm)

significant figure error assumed to be ± 1 in last digi

Note: In this course, all significant quantities will be quoted with an error bar

III Types of error/uncertainty

statistical errors

If you repeat a measurement several times, often you will get slightly different values.
This scatter is the statistical error on the measurement.

average = central value for the measurement.
"spread" = error bar = precision

Systematic errors

This type of error results in an offset in your measurement from the true value that you are trying to measure.

EX: Calibration error

If your ruler is made with cm tick marks that are 11 mm apart (by accident), then you will always measure lengths that are "10%" too short, no matter how many measurements you make.

note: you can try to characterize this 10% shift, but then you need to know it very accurately. (e.g. is it 9%, 11%, or 10% ?)
(or $10.0 \pm 0.4\%$)

$$\boxed{\text{systematic error} = \text{accuracy}}$$

Take home message = design experiment to minimize systematic errors.

IV Statistical / Error Analysis

Consider a number N of measurements $\{x_i\}$ of a quantity x.

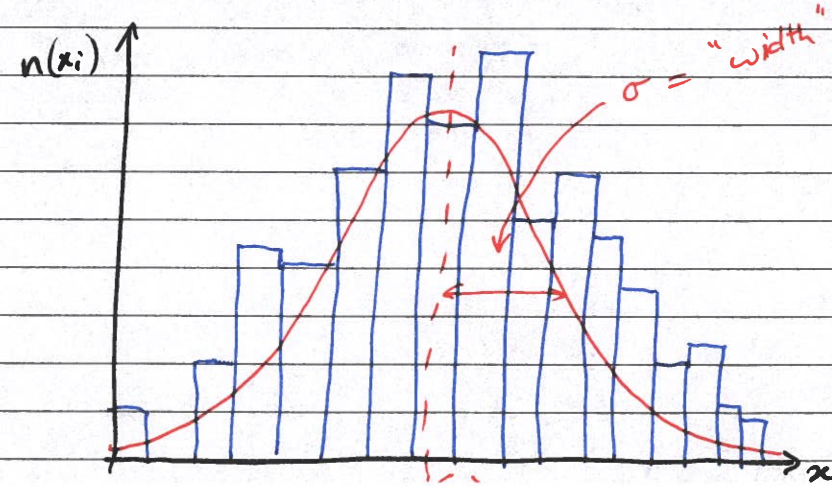
average: average $= \langle x \rangle = \bar{x} = x_{\text{average}}$

$$= \frac{1}{N} \sum_{i=1}^{N} x_i$$

Gaussian distributed errors (covers most statistical errors)

For a large number of measurements, the histogram will be Gaussian:

$$n(x) = \text{Cst}\; e^{-\frac{1}{2}\frac{(x-\langle x \rangle)^2}{\sigma^2}}$$



$$\sigma = \text{"half spread"} = \text{variance} = \frac{1}{\int n(x)\,dx} \int x^2 n(x)\,dx$$

Estimate of σ (for $N =$ large)

$$\sigma_N = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \langle x \rangle)^2} = \text{standard deviation}$$

standard deviation of the mean = error on estimate of the center value. (i.e. average, mean)

$\Delta x =$ error on $\langle x \rangle$

$$= \frac{\sigma_N}{\sqrt{N}} = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \langle x \rangle)^2}{N(N-1)}}$$

measurement of "x" is $\langle x \rangle \pm \Delta x$ "=" x

experimental "="

V Errors on calculated values

A – Single variable function

$$y = f(x) \Rightarrow \Delta y = \frac{\partial f}{\partial x} \Delta x$$

$\hookrightarrow$ error on y : $\pm \Delta y = \pm \left| \frac{\partial f}{\partial x} \right| (\pm \Delta x)$

ex: $y = Ax \Rightarrow \pm \Delta y = |A| (\pm \Delta x)$

B_ Multivariable function

$$F = f(x, y, z, \ldots)$$

$$\Longrightarrow \quad \delta f = \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial y}\,\delta y + \frac{\partial f}{\partial z}\,\delta z + \ldots$$

If the Δx, Δy, and Δz errors are $\underbrace{\text{uncorrelated}}_{\text{i.e. independent}}$,

then

$$\pm\Delta f = \pm\sqrt{\left(\frac{\partial f}{\partial x}\,\Delta x\right)^2 + \left(\frac{\partial f}{\partial y}\,\Delta y\right)^2 + \ldots}$$

$\pm\Delta f$ error on F due to errors $\pm\Delta x$, $\pm\Delta y$, $\pm\Delta z$, ...

Examples

1) $F = x + y \Longrightarrow \pm\Delta F = \pm\sqrt{\Delta x^2 + \Delta y^2}$

2) $F = x - y \Longrightarrow \pm\Delta F = \pm\sqrt{\Delta x^2 + \Delta y^2}$

3) $F = xy \Longrightarrow \pm\Delta F = \pm\underbrace{|F|}_{|xy|}\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$

4) $F = x/y \Longrightarrow \pm\Delta F = \pm\underbrace{|F|}_{|x/y|}\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$