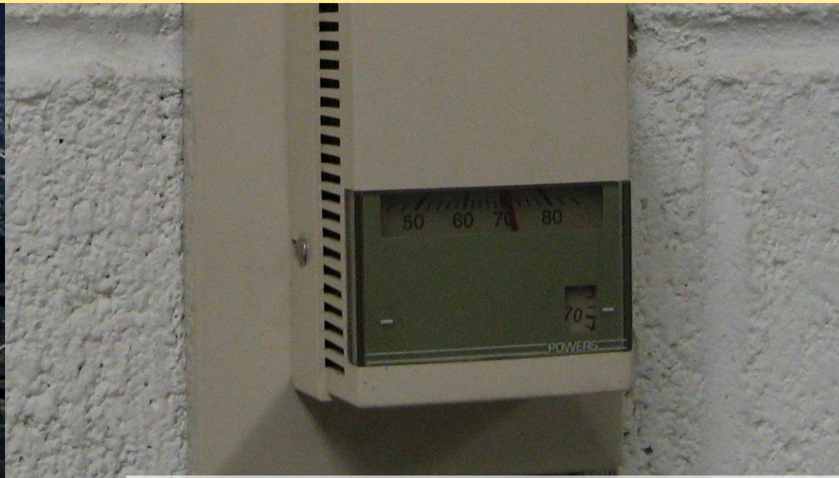




# Feedback Control Theory


Week 11: narrated presentation
Prof. Seth Aubin

# Outline

- Motivation: Why study feedback theory?
- **System Model, Feedback Model**
- *PID feedback control theory*
- **How well does it work?**
- *Back to Fourier space.*
- PID with electronics.

# Why is feedback important?

**Answer:** *Feedback is used in most devices to achieve very high levels of stability to external influences.*

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The idea of using feedback to regulate a systems behavior has been around for a long time (for example, centrifugal governor, circa 1780's)

The quantitative use of **feedback** is one of the primary engineering developments of the 20$^{th}$ century.

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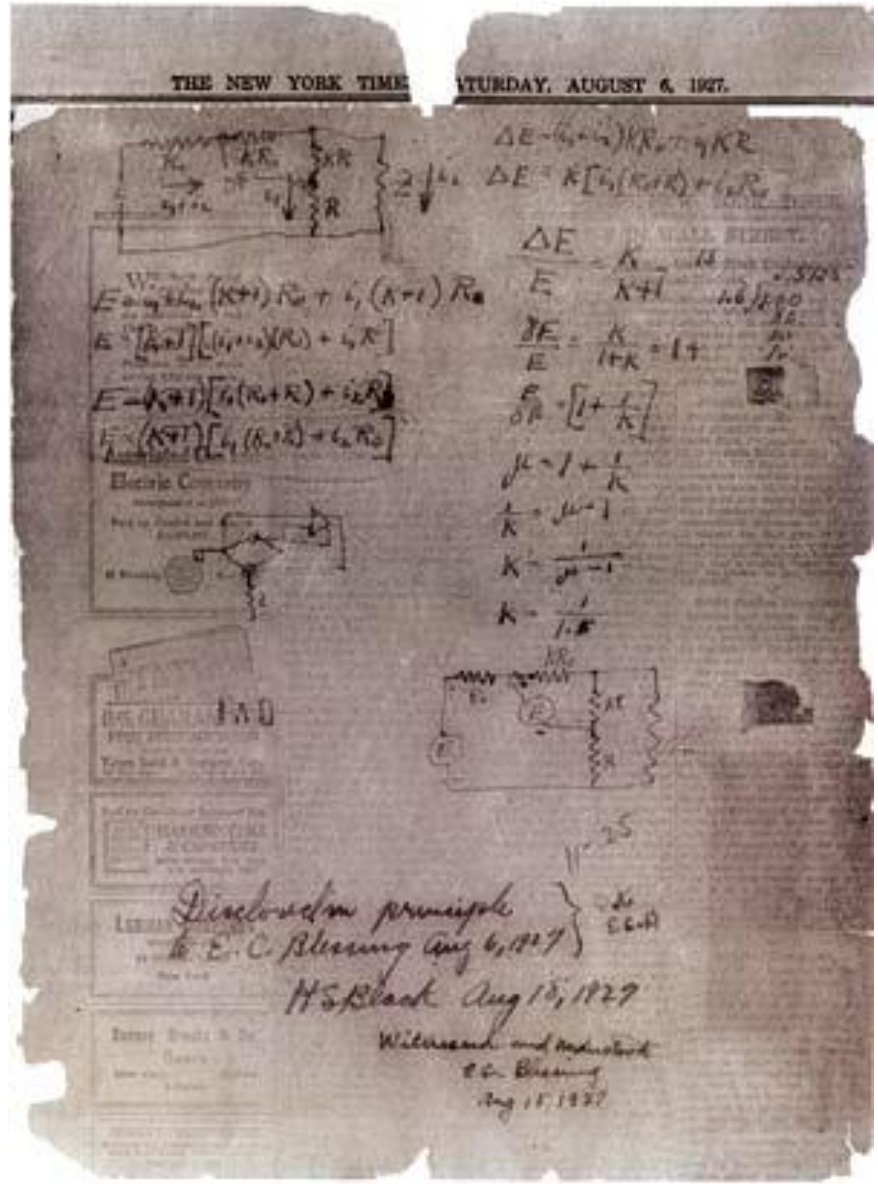
Harold S. Black invented negative feedback to stabilize and linearize the gain of telephone amplifiers (at Bell Telephone Laboratories).

*"Our patent application was treated in the same manner as one for a perpetual-motion machine"*

# A little history …

The original notes by Harold Black which he used to discover the use of feedback for stabilizing amplifiers, while the riding the Manhattan – Staten Island Ferry (1927).




[image from www.wpi.edu]

# Feedback Applications

Engineering:

[image from www.yorku.ca]

F-16 Falcon

[image from www.nellis.af.mil]

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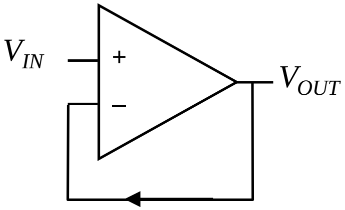


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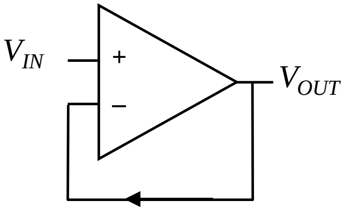
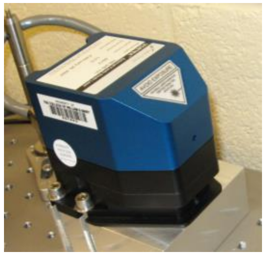
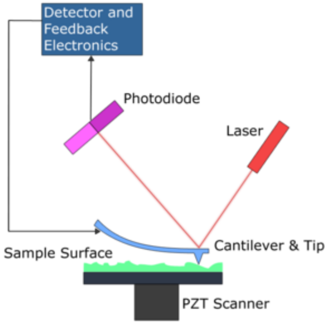



Physics:

CERN: Stochastic Cooling



Laser locking

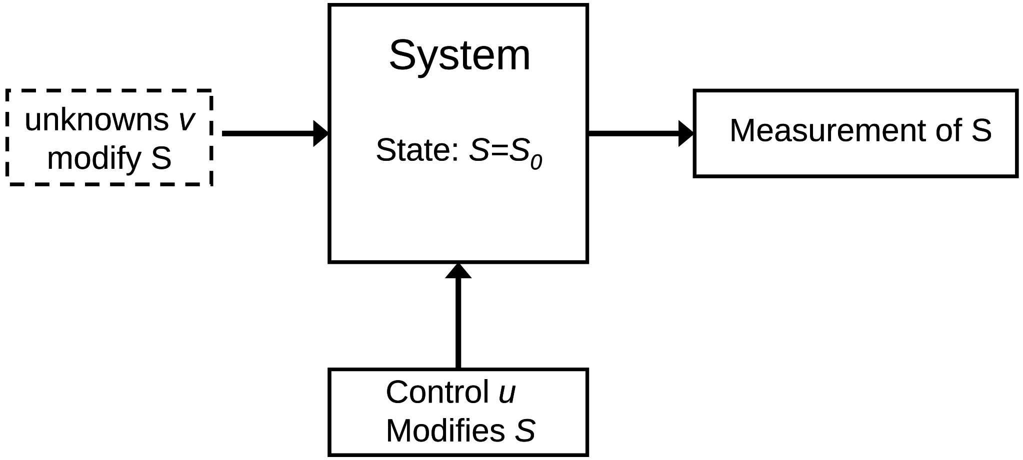



Atomic force microscope
[figure from content.answers.com]

Biology: anything alive.

# Feedback Model

**System with no feedback:**



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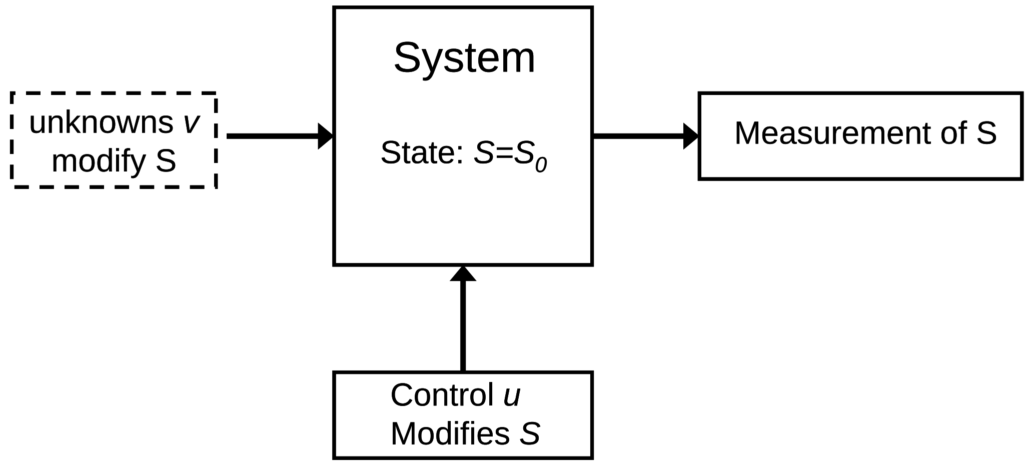
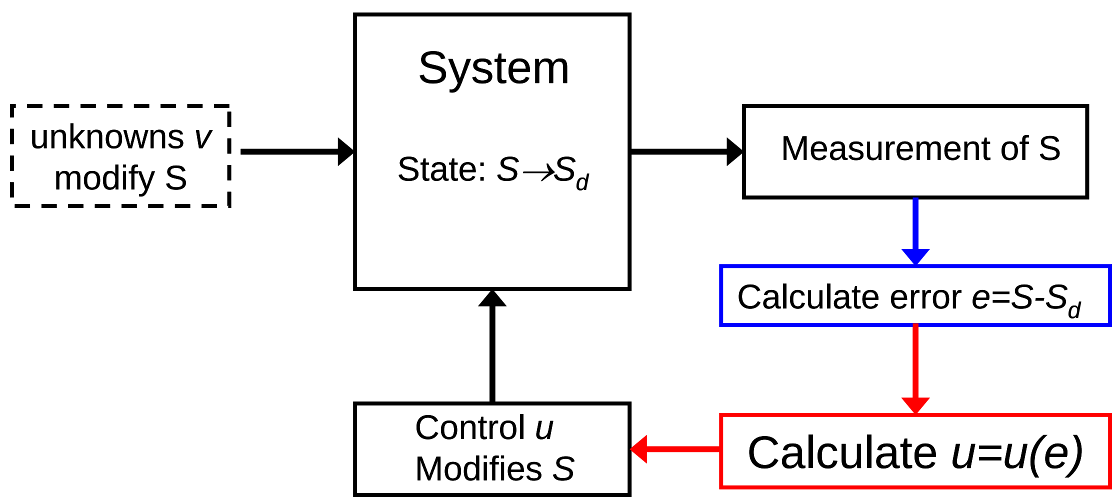
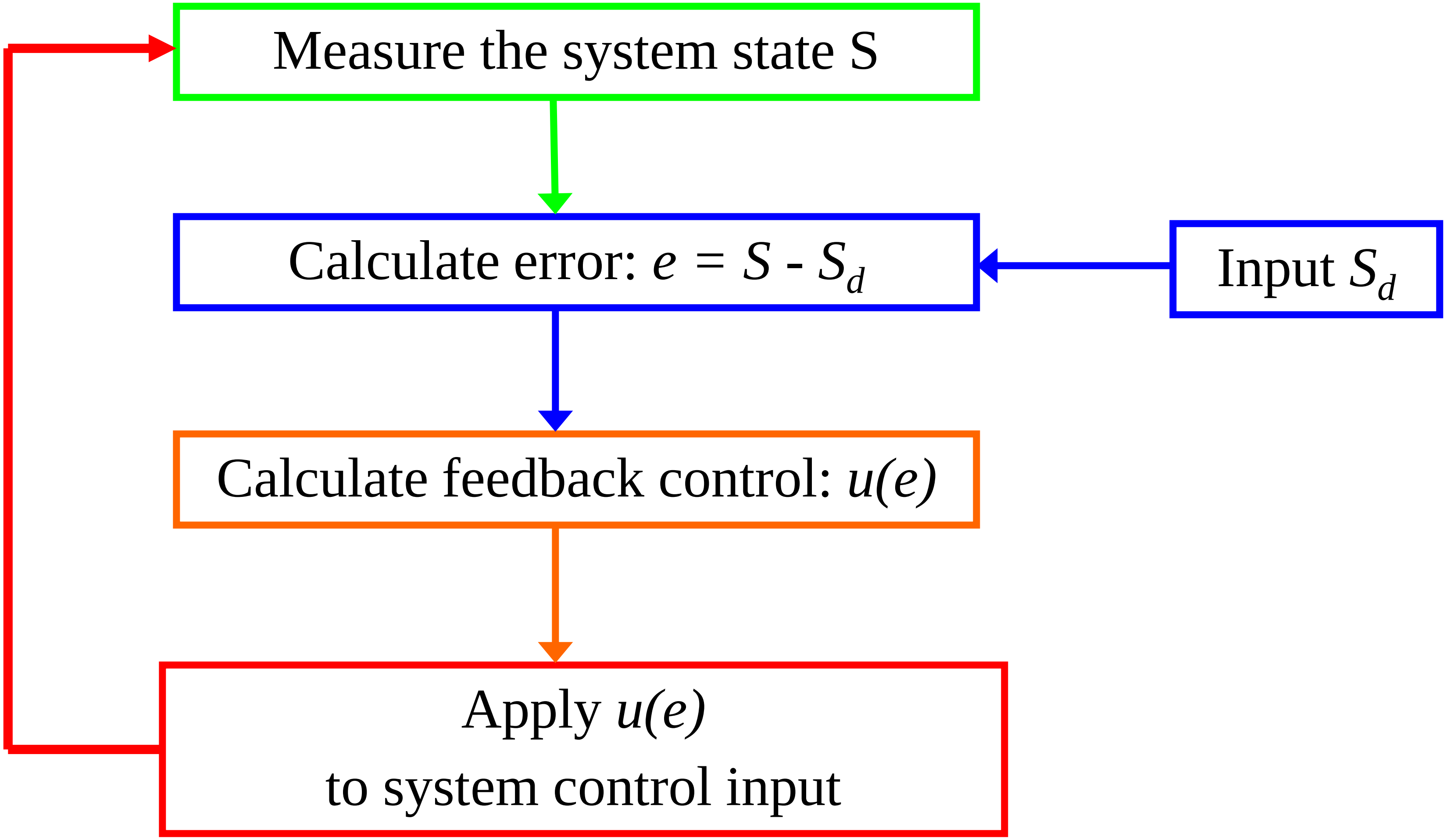



**System with feedback:**

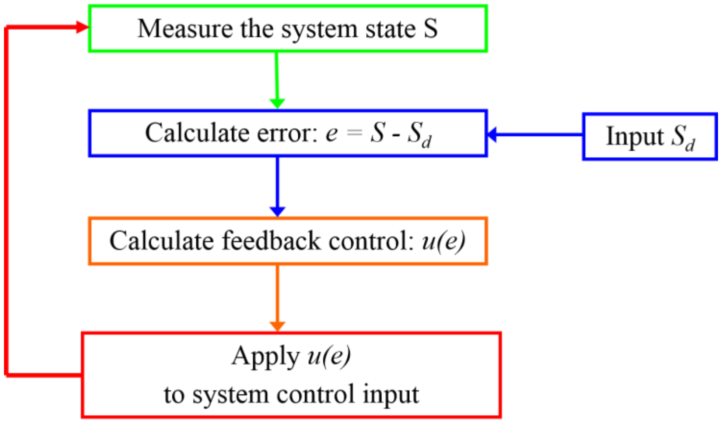



**Claim:** system will converge to system state $S_d$, if $u(e)$ is chosen appropriately.

# Feedback Algorithm



# Feedback Algorithm: Time Evolution



# Feedback Algorithm: Time Evolution

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Basic time evolution equation:

$$\boxed{s(t) + \tau \frac{d\,s(t)}{dt} = s_0 + u(\underbrace{s(t) - s_d}_{\text{error signal } e})}$$

**Question:** What is the best feedback control function $\boldsymbol{u(e)}$ to use ?

How do we find it ?

# PID feedback control
# -- how to calculate *u(e)* --

*Based on a survey of over eleven thousand controllers in the refining, chemicals and pulp and paper industries, 97% of regulatory controllers utilize PID feedback.*

L. Desborough and R. Miller, Honeywell.
Sixth International Conference on Chemical Process Control. *AIChE Symposium Series Number 326* (Volume 98), 2002.

**Proportional gain:** corrects for errors based on the *Present*.

**Integral gain:** corrects for errors based on the *Past*.

**Derivative gain:** corrects for errors based on the anticipated *Future*.

# PID feedback formula

In PID control feedback, we determine $\boldsymbol{u(e)}$ with the following formula:

$$u(e;t) = g_p e(t) + g_i \int_0^t e(t')dt' + g_d \frac{d\,e(t)}{dt}$$

# PID Feedback Differential Equation

*PID feedback formula*

$$u(e;t) = g_p e(t) + g_i \int_0^t e(t')dt' + g_d \frac{d\, e(t)}{dt}$$

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*initial conditions:* $s(t=0) = s_0$

$$\frac{ds}{dt} = 0 \quad at\ t = 0$$

*We apply feedback starting at time t=0.*

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# PID Feedback
# Pure Proportional Gain

$$s(t) + \tau \frac{d\, s(t)}{dt} = s_0 + g_p e(t) + g_i \int_0^t e(t')dt' + g_d \frac{d\, e(t)}{dt}$$

# PID Feedback
# Pure Proportional Gain

$$s(t) + \tau \frac{d\,s(t)}{dt} = s_0 + g_p e(t) + \cancel{g_i \int_0^t e(t')dt'} + \cancel{g_d \frac{d\,e(t)}{dt}}$$

For pure proportional gain:

$$\boxed{s(t) + \tau \frac{d\,s(t)}{dt} = s_0 + g_p e(t)}$$

# PID Feedback
# Pure Proportional Gain: SOLUTION

$$\boxed{s(t) + \tau \frac{d\,s(t)}{dt} = s_0 + g_p e(t)}$$

Solution: $$s(t) = \left[s_0 - \frac{s_0 - g_p s_d}{1 - g_p}\right] e^{-\frac{(1-g_p)}{\tau}t} + \frac{s_0 - g_p s_d}{1 - g_p}$$

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If $g_p > 1 \Rightarrow \frac{\tau}{1-g_p} < 0 \Rightarrow$ positive feedback … exponential & system diverge.

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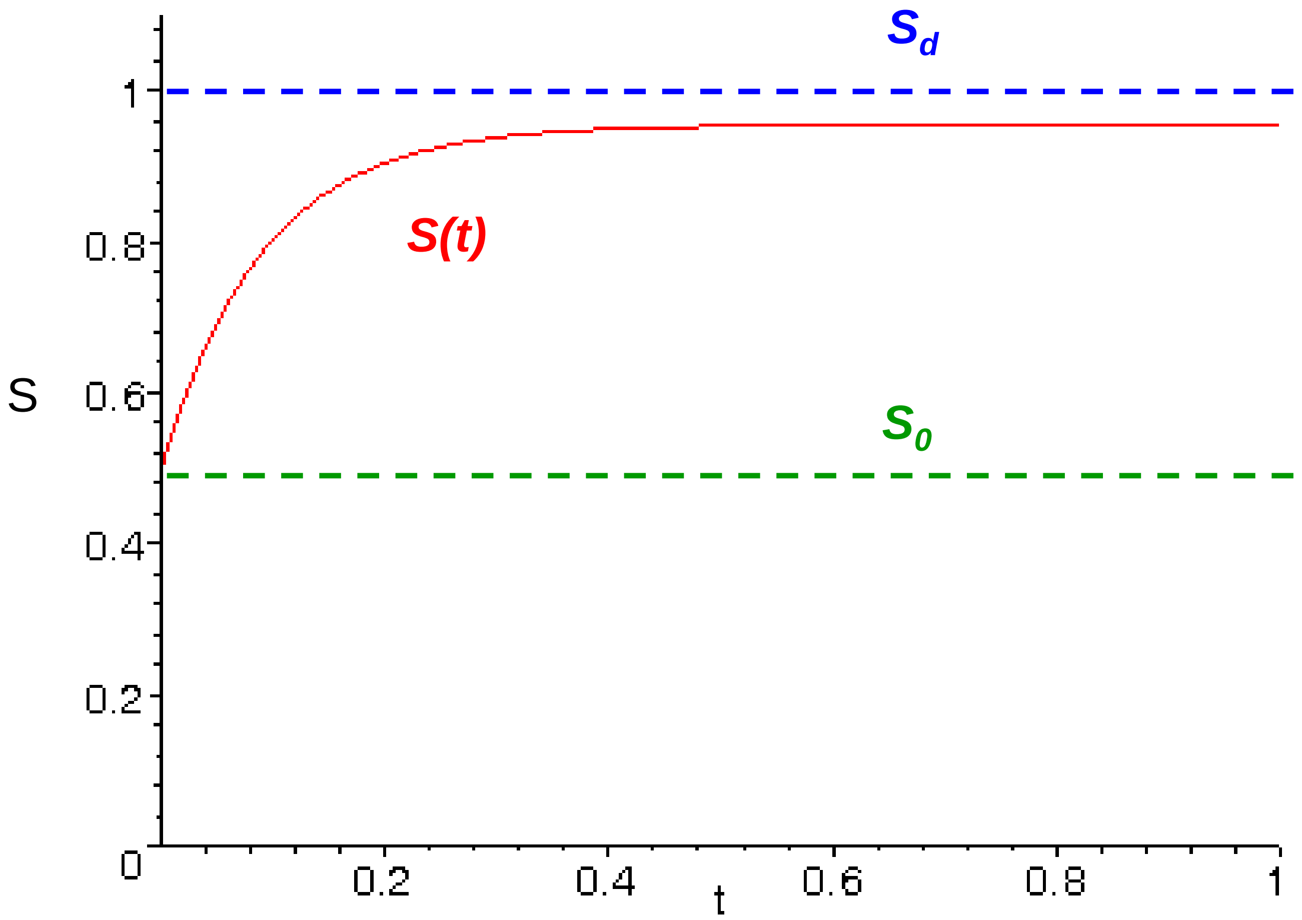
If $g_p < 1 \Rightarrow \frac{\tau}{1-g_p} > 0 \Rightarrow$ negative feedback … exponential dies out.

$g_p < 0 \;\Longrightarrow$ *True negative feedback.*

# How well does it work?

Proportional Gain:



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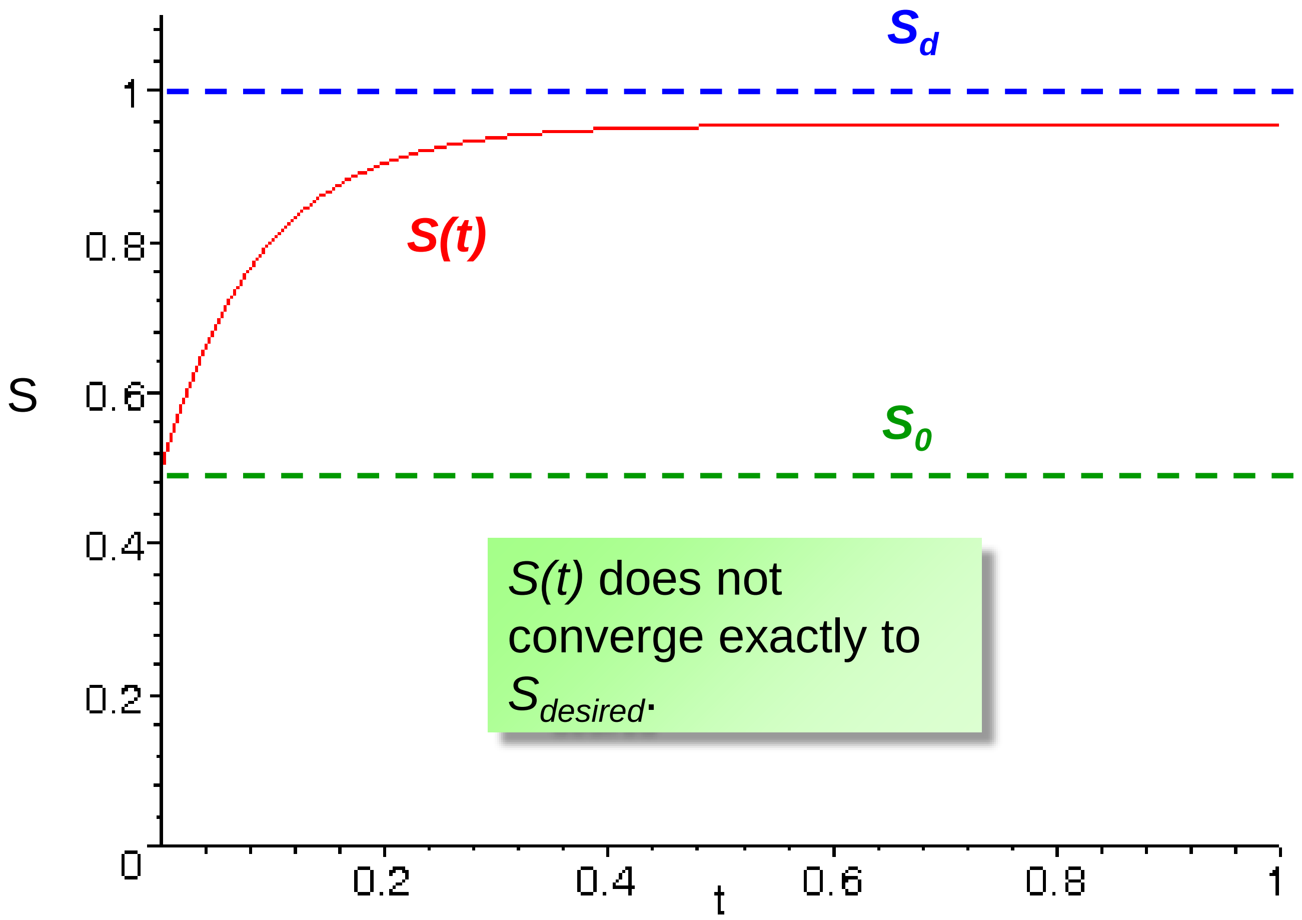


$S(t)$ does not converge exactly to $S_{desired}$.

# Pure <u>Proportional</u> Gain Conclusions

Steady state value: $$\boxed{S_{steady\ state} = \frac{S_0 - g_p S_d}{1 - g_p}}$$

Time response of system:
*(due to feedback)*

$$\boxed{\tau_{new} = \frac{\tau}{1 - g_p}}$$

**<u>Main Conclusions</u>**

- *S* does not converge exactly to $S_{desired}$ (there's an offset).
- For $g_p$ → very negative, we have: - System converges quicker to steady state.
  - System converges closer to $S_{desired}$.

# PID Feedback
# Proportional-Integral Gain

Master equation:

$$s(t) + \tau \frac{d\,s(t)}{dt} = s_0 + g_p e(t) + g_i \int_0^t e(t')dt' + g_d \frac{d\,e(t)}{dt}$$

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$$\frac{ds}{dt} + \tau \frac{d^2 s}{dt^2} = g_p \frac{ds}{dt} + g_i (s - s_d)$$

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$$\frac{ds}{dt} + \tau \frac{d^2 s}{dt^2} = g_p \frac{ds}{dt} + g_i (s - s_d)$$

regroup terms

$$\tau \frac{d^2 s}{dt^2} + (1 - g_p) \frac{ds}{dt} - g_i\, s = -g_i\, s_d$$

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# Proportional-Integral Gain: SOLUTION

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2nd order diff. eq. with constant coefficients

*(inhomogenous)*

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Standard Solution !

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$$s(t) = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t} + s_d$$

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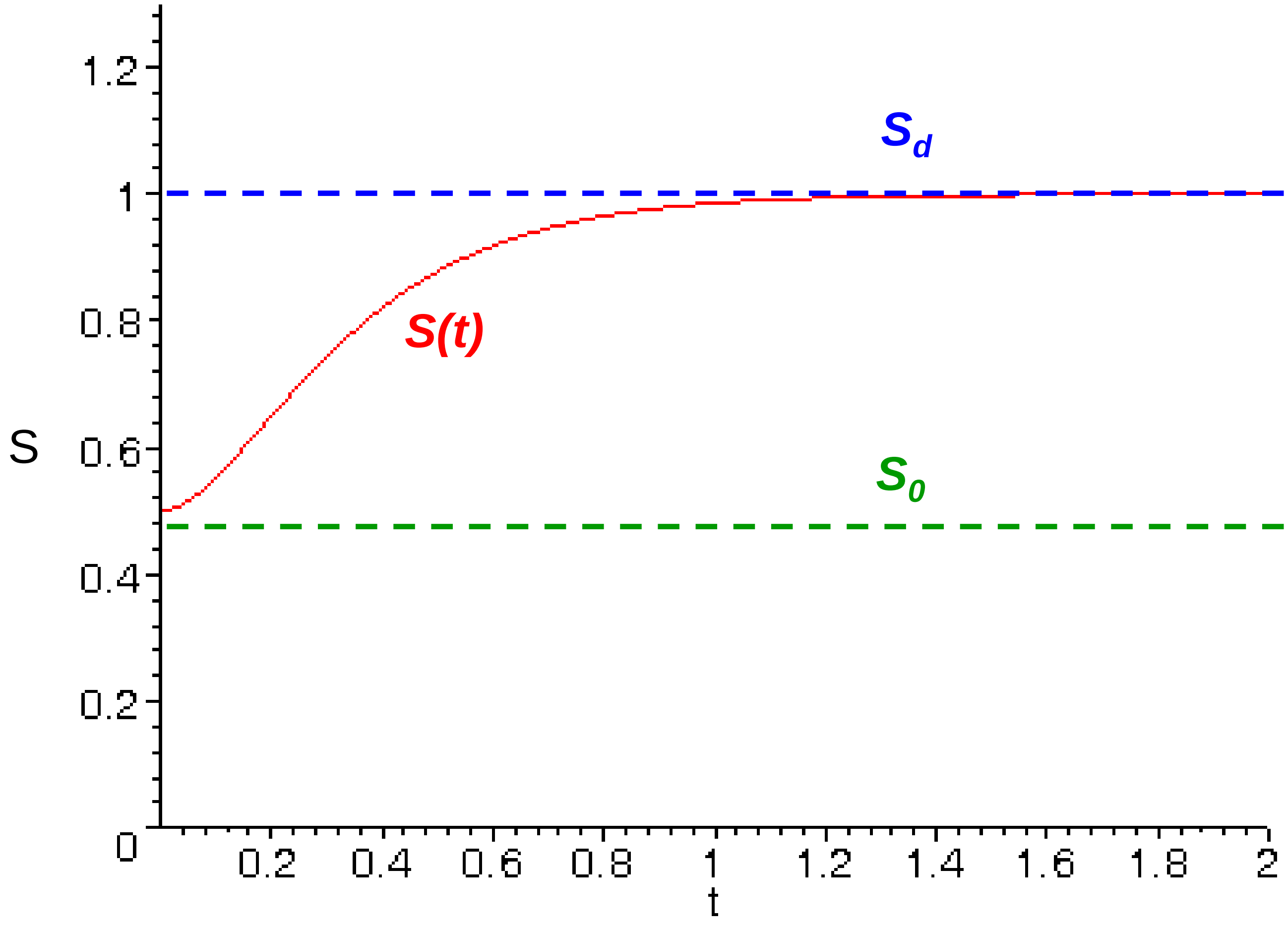
and $A_\pm = \mp\left(\dfrac{\lambda_\mp}{\lambda_- + \lambda_+}\right)(s_d - s_0)$

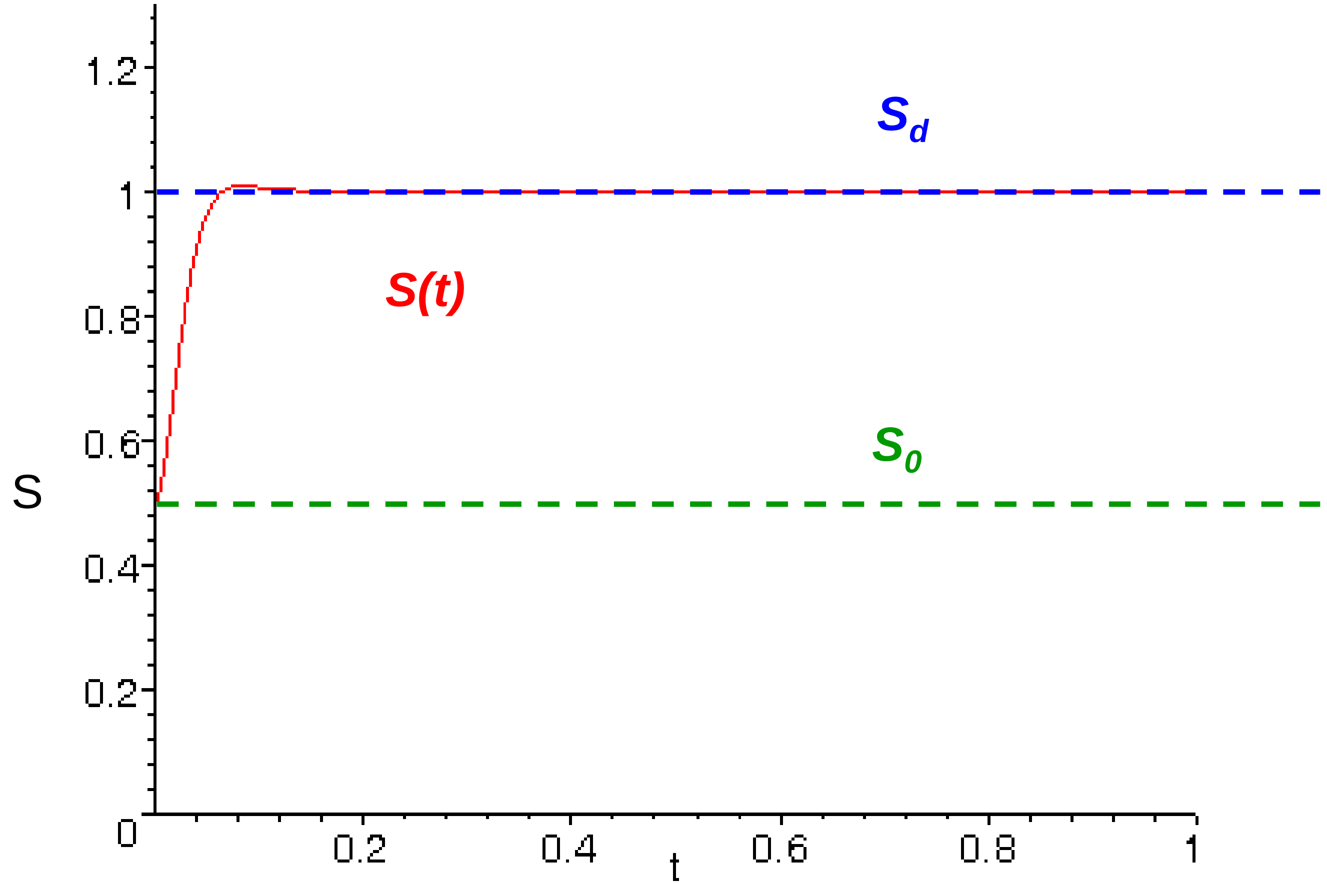
# How well does it work?

Proportional – Integral Gain: $g_p$=-10, $g_i$=-30, $\tau$=1



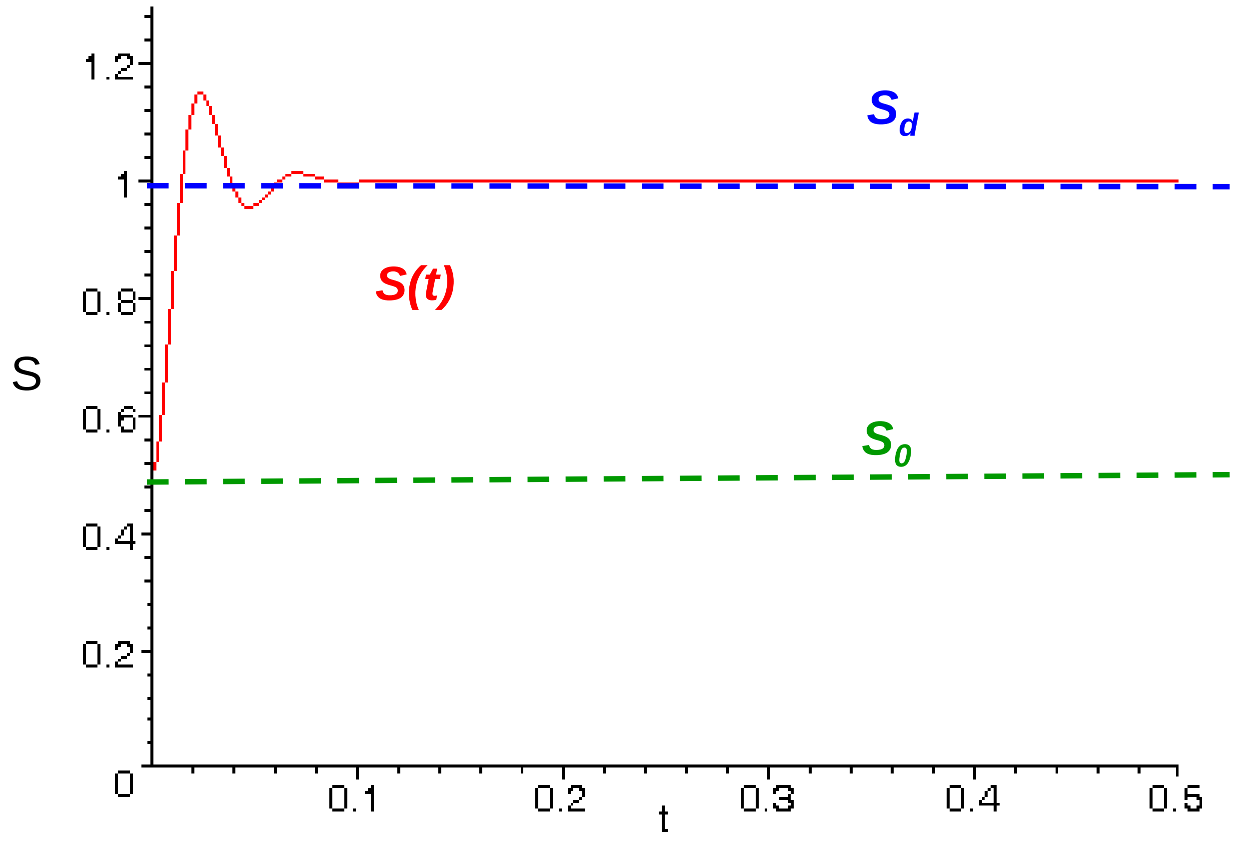
# How well does it work?

Proportional – Integral Gain: $g_p$=-100, $g_i$=-4000, $\tau$=1



# How well does it work?

Proportional – Integral Gain: $g_p$=-100, $g_i$=-20,000, $\tau$=1



# Proportional-Integral Feedback
# CONCLUSIONS

$$s(t) = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t} + s_d$$

with $\lambda_\pm = \dfrac{-(1-g_p) \pm \sqrt{(1-g_p)^2 + 4\tau g_i}}{2\tau}$ and $A_\pm = \mp\left(\dfrac{\lambda_\mp}{\lambda_- + \lambda_+}\right)(s_d - s_0)$

- $s(t)$ converges to $s_d$ → very good !
- $\lambda_+ < 0$ and $\lambda_- < 0$ for the system to converge.
- If $\lambda_\pm$ has an imaginary part, then the system will have damped oscillations.
- We want $\lambda_+$ and $\lambda_-$ to be as negative as possible, i.e. $g_p \ll 0,\ g_i \ll 0$.

# Fourier space: Noise suppression

**Idea:** Noise can cause the system to “blow up”, i.e. oscillate wildly, if the system is unstable at some frequency.

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The solution now becomes: $s(t) = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t} + s_d + B s_n e^{i\omega t}$

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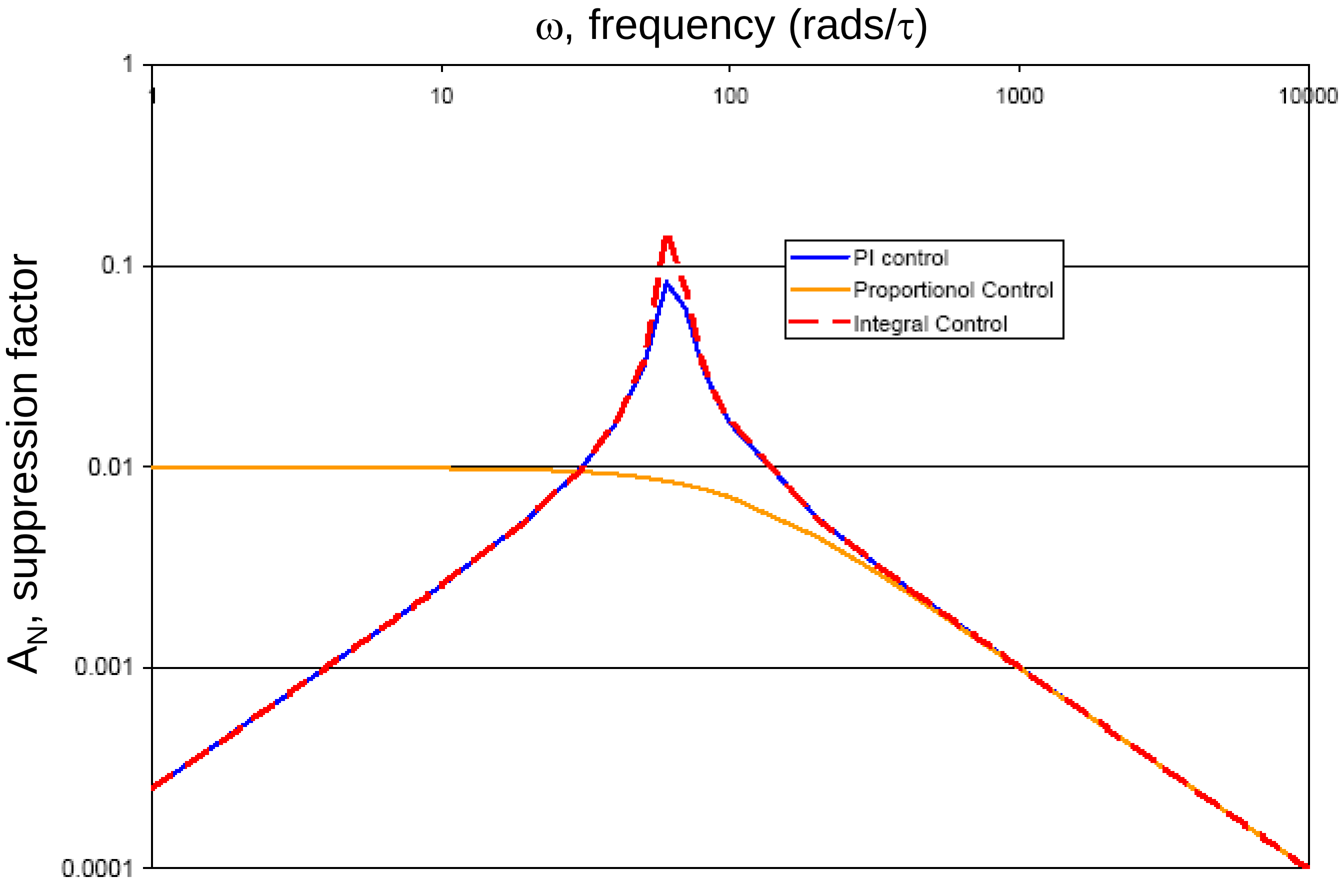
with $B = \frac{i\omega}{i\omega(1-g_p)-\tau\omega^2-g_i}$

Noise Suppression Factor: $\boxed{A_{NS} = |B| = \frac{\omega}{(\tau\omega^2+g_i)^2+(1-g_p)^2\omega^2}}$

# Fourier space: Noise suppression

Proportional – Integral Gain: $g_p=-100,\ g_I=-4000,\ \tau=1$



# Fourier space: Noise suppression

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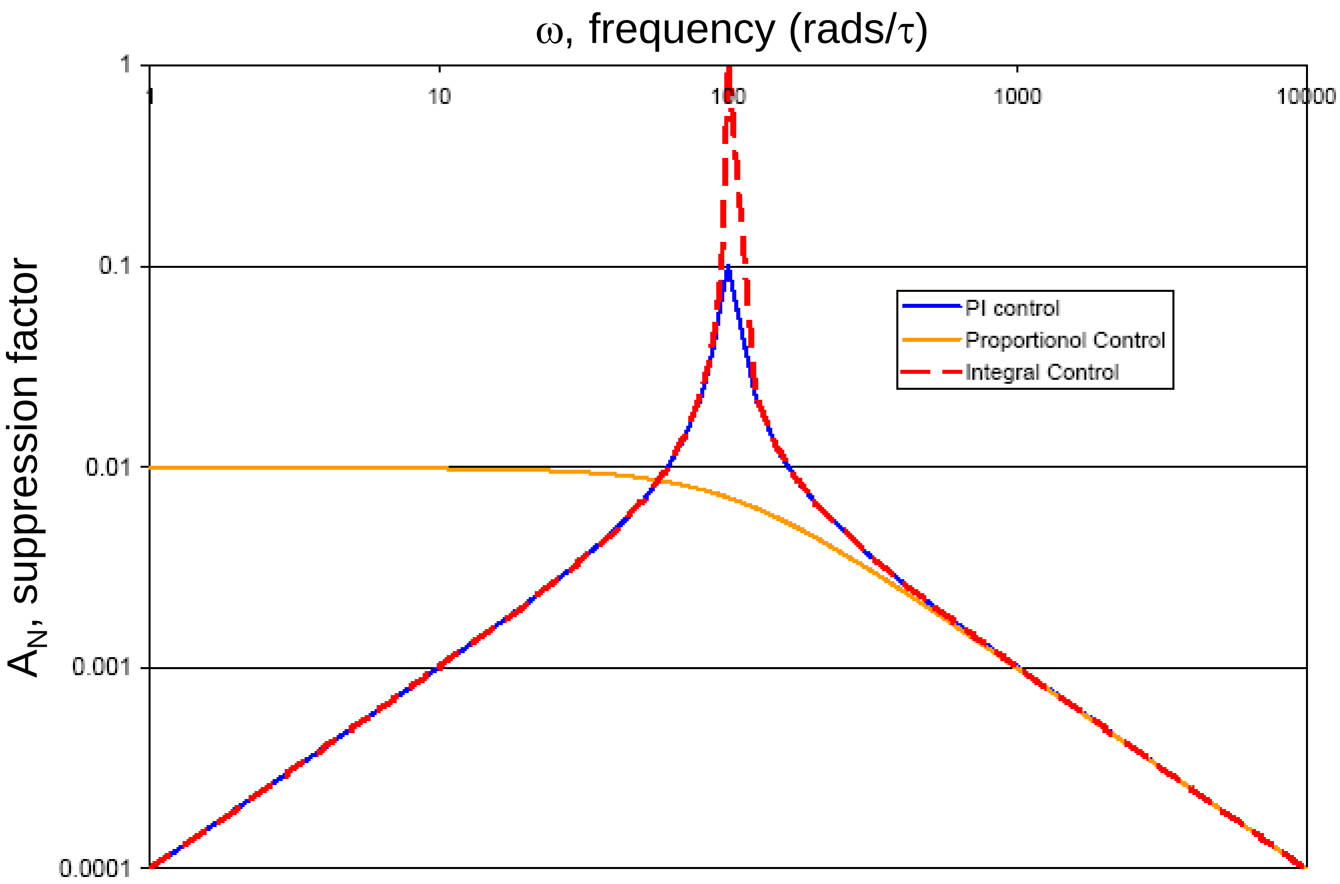
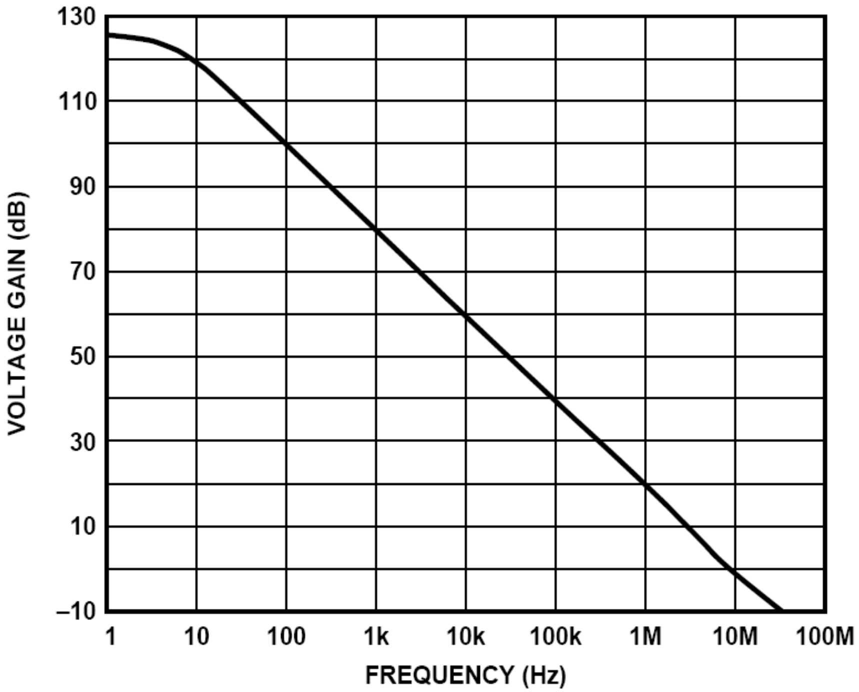


# Reality: Gain is not flat




[From the OP27 datasheet]
*(good quality op-amp)*