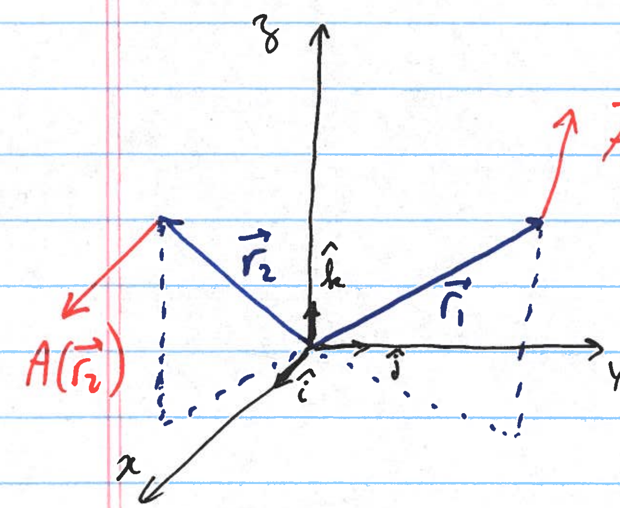
Wednesday, January 27, 2021

Basic Review of Vector Calculus

I Vector Fields (cartesian coordinates)

Definition (physics): Every point in space (3D) has a vector associated with it (typically "abstract", non-spatial, 3D).



$$\vec{A}(\vec{r}_1) = \left(A_x(\vec{r}_1),\ A_y(\vec{r}_1),\ A_z(\vec{r}_1)\right)$$

$$= \left(A_x(x,y,z),\ A_y(x,y,z),\ A_z(x,y,z)\right)$$

$$= A_x(\vec{r}_1)\,\hat{i} + A_y(\vec{r}_1)\,\hat{j} + A_z(\vec{r}_1)\,\hat{k}$$

$$\hat{i} \leftarrow \hat{x} \qquad \hat{j} \leftarrow \hat{y} \qquad \hat{k} \leftarrow \hat{z}$$

Generally/Typically, the vector field components (A_x, A_y, A_z) are continuous and differentiable functions of the cartesian coordinates $\vec{r} = (x, y, z)$ [and time t]

not often in this course

II Vector products of two vectors $\vec{A}$ & $\vec{B}$ or vector fields

A- Scalar product ("dot" product) = number

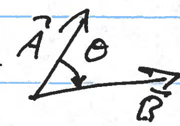
$$\underbrace{\vec{A}\cdot\vec{B}} = A_x B_x + A_y B_y + A_z B_z = \vec{B}\cdot\vec{A}$$

(commutative)

$$= |\vec{A}|\,|\vec{B}|\cos\underbrace{\theta_{AB}}$$

$\vec{A}(\vec{r}_1)\cdot\vec{B}(\vec{r}_1)$

angle between $\vec{A}$ & $\vec{B}$



note: if you find $\vec{A}(\vec{r}_1)\cdot\vec{B}(\vec{r}_2)$, then there is probably a mistake! not local!

Vector magnitude (i.e. "length"): $A = |\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}$

B- Vector product ("cross" product) = vector

$$\vec{A}\times\vec{B} = \begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix} \times \quad = -\vec{B}\times\vec{A}$$

$$= (A_y B_z - A_z B_y,\; A_z B_x - A_x B_z,\; A_x B_y - A_y B_x)$$

$$= |\vec{A}|\,|\vec{B}|\sin\theta_{AB}\;\hat{n}$$

unit vector perpendicular to $\vec{A}$-$\vec{B}$ plane in direction given by right hand rule

III Vector field "derivatives"

A. Gradient

single component vector field

The gradient of a $\underbrace{\text{scalar field}}$ $f(x,y,z)$ is the vector:

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$= \partial_x f \, \hat{x} + \partial_y f \, \hat{y} + \partial_z f \, \hat{z}$$

direction of $\vec{\nabla} f$: $\vec{\nabla} f$ points in the direction of maximum increase of the function $f(x,y,z)$

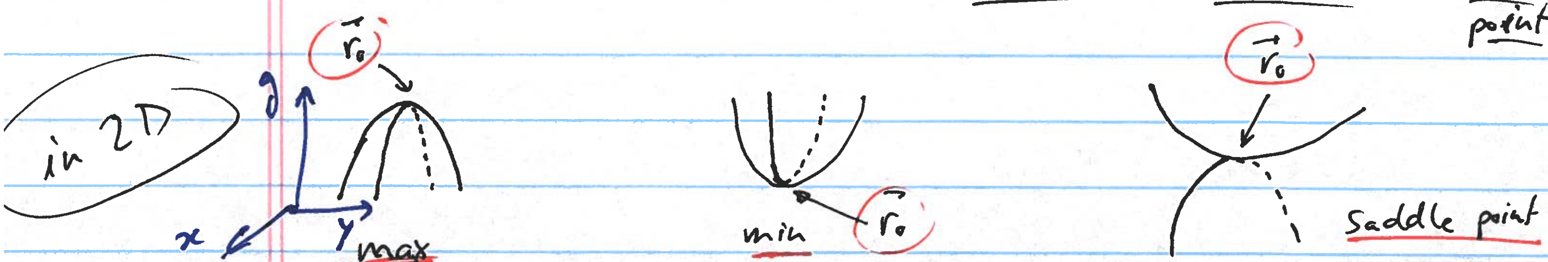
magnitude of $\vec{\nabla} f$: $|\vec{\nabla} f|$ gives the slope along the direction of maximum increase.

Del operator $\vec{\nabla}$: $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$
(vector operator)

Q: if $\vec{\nabla} g(x,y,z) = \vec{0} = (0,0,0)$

for $\vec{r}_0 = (x_0, y_0, z_0)$ then what can we say about $\vec{r}_0$

A: $\vec{r}_0$ is either a local maximum, minimum, or saddle point



example: Calculate $\vec{\nabla}\frac{1}{r}$ $\left[\text{recall: } r = |\vec{r}| = \sqrt{x^2+y^2+z^2}\right]$

$$\vec{\nabla}\frac{1}{r} = (\partial_x, \partial_y, \partial_z)\frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2}$$

$$= \left(\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}}, \frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+z^2}}, \frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \left(-\frac{1}{\cancel{2}}\frac{\cancel{2}x+0+0}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}}\right)$$

note: $(x^2+y^2+z^2)^{3/2} = (r^2)^{3/2} = r^3 = r^2 r$

$$= \left(\frac{-x}{r^2 r}, \frac{-y}{r^2 r}, \frac{-z}{r^2 r}\right)$$ but $\vec{r} = (x, y, z)$

$$= -\frac{\vec{r}}{r^2 r} = -\frac{\hat{r}}{r^2}$$

$$\Rightarrow \boxed{\vec{\nabla}\frac{1}{r} = -\frac{\hat{r}}{r^2}}$$