Monday, January 31, 2021

III Vector Field Derivatives (continued)

B. Divergence

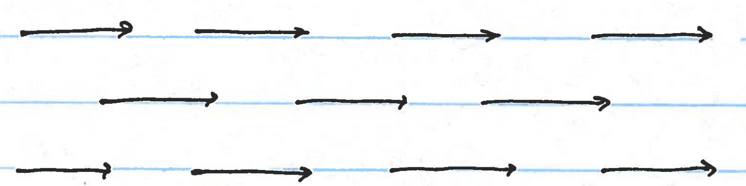
The divergence of a <u>vector field</u> $\vec{A}(\vec{r})$ is a <u>scalar</u>:

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

<u>example 1:</u> $\vec{A}(\vec{r}) = v_0\,\hat{x} = (v_0, 0, 0)$

i.e. "constant flow", if you think of the vector field as a fluid velocity map.

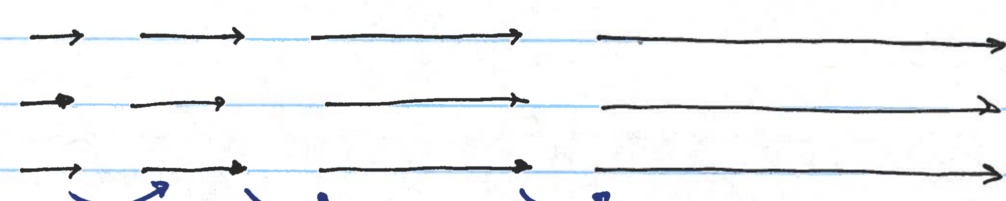
$$\vec{\nabla}\cdot\vec{A} = \cancelto{=0}{\frac{\partial}{\partial x}v_0} + \underbrace{\frac{\partial}{\partial y}0}_{=0} + \underbrace{\frac{\partial}{\partial z}0}_{=0} = 0$$



<u>example 2:</u> $\vec{A}(\vec{r}) = (ax, 0, 0)$

i.e. "increasing flow" (or "accelerating" flow)

$$\vec{\nabla}\cdot\vec{A} = \cancelto{=a}{\frac{\partial}{\partial x}ax} + \underbrace{\frac{\partial}{\partial y}0}_{=0} + \underbrace{\frac{\partial}{\partial z}0}_{=0} = a$$



if fluid flow increases (velocity increases) then fluid must have been <u>created</u> (at constant density)

Divergence is positive when "fluid" is created/sourced

Divergence is negative when "fluid" is eliminated/"sunk."

C – Curl

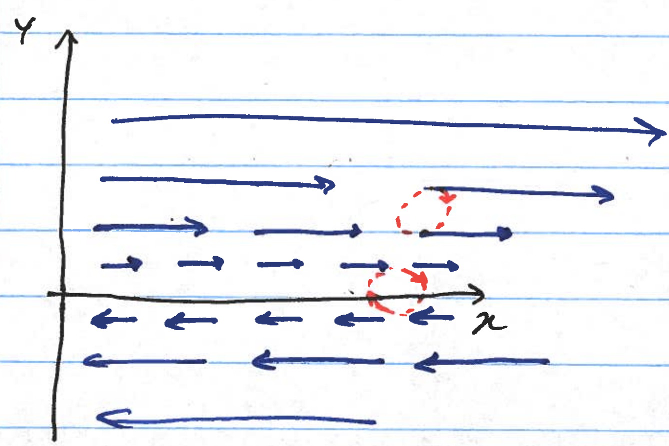
The curl of a vector field $\vec{A}(\vec{r})$ is a vector

$$\vec{\nabla}\times\vec{A} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(A_x, A_y, A_z\right)$$

$$= \left(\partial_y A_z - \partial_z A_y,\ \partial_z A_x - \partial_x A_z,\ \partial_x A_y - \partial_y A_x\right)$$

Example 1: Both $\underbrace{\vec{A} = v_0\hat{x}}_{\vec{\nabla}\times\vec{A}=0}$ and $\underbrace{\vec{B} = ax\,\hat{x}}_{\vec{\nabla}\times\vec{B}=0}$ have curl = 0.

Example 2: $\vec{C} = cy\,\hat{x} = (cy, 0, 0)$



$$\vec{\nabla}\times\vec{C} = (0,\ 0-0,\ 0-c)$$
$$= -c\,\hat{z} \neq 0$$

The curl is non-zero when the "fluid flow" of the vector field can "rotate" (partially) a finite particle e.g. "cheerio"

D. Laplacian

Consider a scalar field $f(\vec{r})$. The Laplacian of f is given by

$$\nabla^2 f = \Delta f = \underbrace{\vec{\nabla}\cdot(\vec{\nabla} f)}_{\text{"div-grad"}} = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The Laplacian of a vector field $\vec{A}(\vec{r})$ is given by

$$\nabla^2 \vec{A} = \Delta \vec{A} = (\vec{\nabla}\cdot\vec{\nabla})\vec{A} = \left(\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z\right)$$

⚠ $\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) \neq \nabla^2\vec{A}$ and is not used much on its own.

E. Other second derivatives

$f(\vec{r})$ = scalar field

$\vec{A}$ = vector field

Div-Curl: $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

Some algebra (recall: $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f$)

Some algebra

Curl-grad: $\vec{\nabla}\times(\vec{\nabla} f) = \vec{0}$

Curl-curl: $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

Some algebra

F. Convective Derivative (or material derivative)

[not important for PHYS 401, but may show up in PHYS 402 and fluid dynamics plasma physics]

$$\frac{d}{dt} f(\underbrace{x, y, z}_{\vec{r}=\vec{r}(t)}; t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\underbrace{\frac{\partial x}{\partial t}}_{v_x} + \frac{\partial f}{\partial y}\underbrace{\frac{\partial y}{\partial t}}_{v_y} + \frac{\partial f}{\partial z}\underbrace{\frac{\partial z}{\partial t}}_{v_z}$$

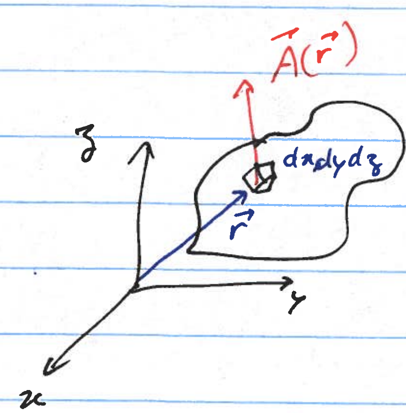
$$= \frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f$$

For a vector field: $\frac{d}{dt}\vec{A} = \frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A}$

$$\Rightarrow \boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})}$$

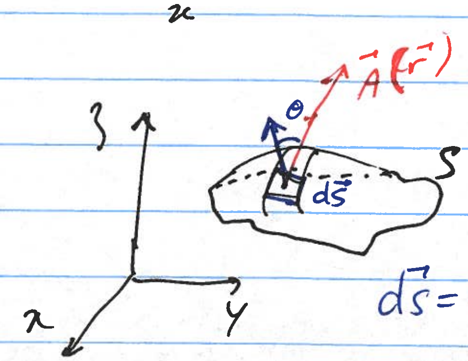
IV Integrals

Volume: $\int_V \vec{A}(\vec{r})\,\underbrace{dx\,dy\,dz}_{dV = d^3r} = \text{vector}$



Surface:

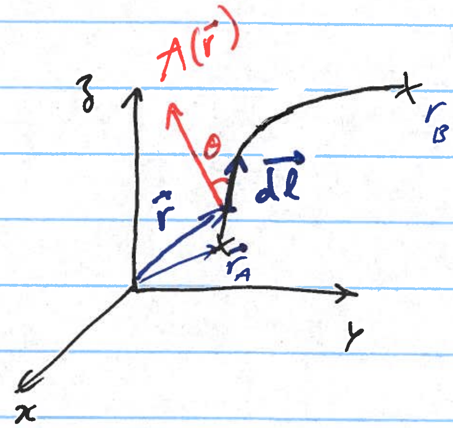
$$\int_S \vec{A}(\vec{r})\cdot d\vec{s} = \text{scalar}$$



$d\vec{s} = ds\cdot\hat{n}$ ← normal to surface note: For a closed surface (e.g. balloon, potato skin) $\underline{\underline{d\vec{S}}}$ sticks out by convention.

$\hookrightarrow \quad \int_S \rightarrow \oint_S$

Line:

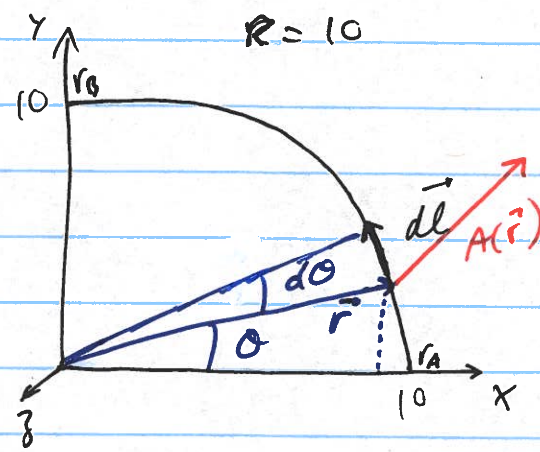


$$\int_{r_A \text{ path}}^{r_B} \vec{A}(\vec{r}) \cdot d\vec{l} = \text{scalar}$$

closed path $\displaystyle\oint_{\text{path}} \vec{A} \cdot d\vec{l} = \text{scalar}$

example: quarter circle arc

step 1: parametrize the path (using θ)



$\vec{r} = (R\cos\theta, R\sin\theta, 0)$

$\vec{dl} = \hat{l}\, dl$

$\hookrightarrow$ with $dl = R\, d\theta$

$\hat{l} = (-\sin\theta, \cos\theta, 0) \longleftarrow$ norm $= 1$

$(\sin^2\theta + \cos^2\theta = 1)$

$\Rightarrow \vec{dl} = (-R\sin\theta\, d\theta, R\cos\theta\, d\theta, 0) = (-10\sin\theta\, d\theta, 10\cos\theta\, d\theta, 0)$

Easy Example with $\vec{A} = (5, 3, -1) = 5\hat{x} + 3\hat{y} - \hat{z}$

$$\int_{r_A=(10,0,0)}^{r_B(0,10,0)} \vec{A}\cdot d\vec{l} = \int_{\theta=0}^{\theta=\pi/2} \vec{A}\cdot d\vec{l} = \int_0^{\pi/2}(-50\sin\theta\, d\theta + 30\cos\theta\, d\theta + 0)$$

path = 1/4 arc circle
radius = 10

$$= \underbrace{50\cos\theta\Big|_0^{\pi/2}}_{0-1} + \underbrace{30\sin\theta\Big|_0^{\pi/2}}_{1-0}$$

$$= -50 + 30$$

$$= -20$$

V Integral Theorems

A. Gradient theorem

Consider a vector field $\vec{A}(\vec{r}) = \vec{\nabla} f(\vec{r})$, then

$$\int_{\vec{r}_A}^{\vec{r}_B} \vec{A}\cdot d\vec{l} = \boxed{\int_{\vec{r}_A}^{\vec{r}_B} (\vec{\nabla} f(\vec{r}))\cdot d\vec{l} = f(\vec{r}_B) - f(\vec{r}_A)}$$

path P

$f(\vec{r})$ is called the potential of $\vec{A}$

corollary: the integral depends only on the end points $\vec{r}_A$ & $\vec{r}_B$, not the path.

note: $\oint_{\text{closed path } P \text{ (loop)}} \vec{\nabla} f \cdot d\vec{l} = 0$

example: In a DC electric circuit, the voltage ~~drop~~ difference betwee A & B does not depend on the shape of the wires.