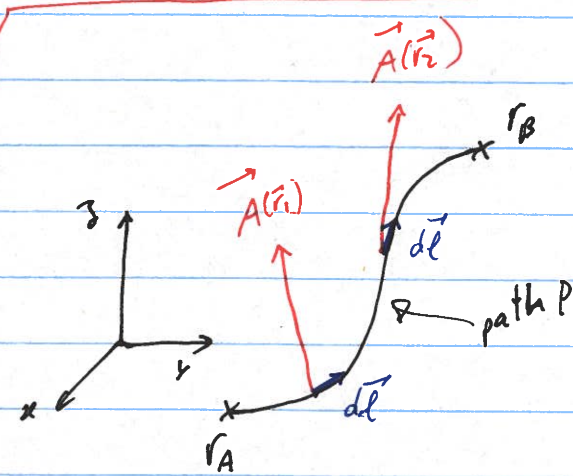
Wednesday, February 3, 2021

# IV Integral Theorems

## A- Gradient Theorem

Consider a vector field $\vec{A} = \vec{\nabla} f(\vec{r})$, then

$$\int_{\vec{r}_A \text{ path } P}^{\vec{r}_B} \vec{A} \cdot d\vec{\ell} = \boxed{\int_{\vec{r}_A}^{\vec{r}_B} (\vec{\nabla} f(\vec{r})) \cdot d\vec{\ell} = f(\vec{r}_B) - f(\vec{r}_A)}$$




note: $f(\vec{r})$ is called the potential of $\vec{A}(\vec{r})$.

Corollary: The integral depends only on the endpoints $\vec{r}_A$ & $\vec{r}_B$, not the path.

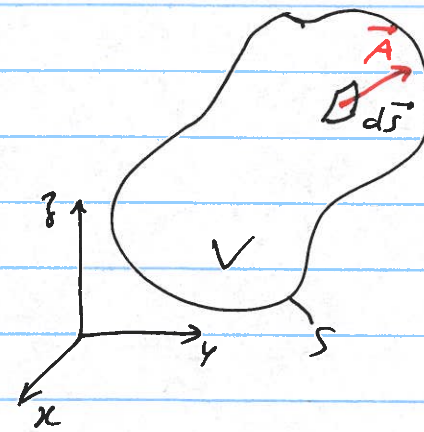
$$\hookrightarrow \oint_{\text{closed path } P \text{ (loop)}} (\vec{\nabla} f) \cdot d\vec{\ell} = 0$$

Example: In a DC electric circuit, the voltage difference between ~~#~~ leads A & B does not depend on the shape of the wires.

## B_ Divergence theorem / Gauss's theorem ["Green's theorem"]

Consider a vector field $\vec{A}(\vec{r})$ and a volume $V$ with surface $S$, then

$$\boxed{\int_V (\vec{\nabla}\cdot\vec{A})\,dV = \oint_S \vec{A}\cdot d\vec{s}}$$

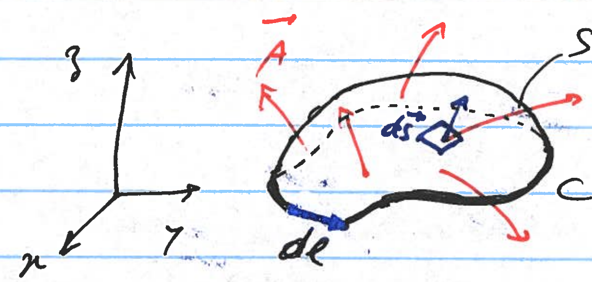


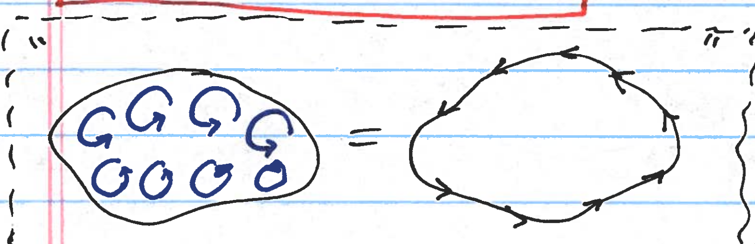

" $\int$ sources/sinks in $V$ = $\oint$ flow out/in through $S$ "

## C_ Stokes's theorem

Consider a vector field $\vec{A}(\vec{r})$ and a surface $S$ with a bounding line $C$, then

$$\boxed{\int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{s} = \int_C \vec{A}\cdot d\vec{\ell}}$$



note 1: $\int_S (\vec{\nabla}\times A)\cdot d\vec{s}$ does not depend on the surface $S$, but only the boundary $C$.

$\vec{\nabla}\times\vec{A}$ = local circulation (i.e. "rotation") of the field $\vec{A}$ (if thought of as fluid flow)

note 2: $\oint_S (\vec{\nabla}\times\vec{A})\cdot d\vec{s} = 0$ for a closed surface $S$

## D- Integration by parts

$f(\vec{r})$ = scalar field

$\vec{A}(\vec{r})$ = vector field

$V$ = volume with surface $S$

$$\int_V \underbrace{f(\vec{\nabla}\cdot\vec{A})}\, d^3r$$

from front of book: $\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$

$$\Rightarrow \; f(\vec{\nabla}\cdot\vec{A}) = \vec{\nabla}\cdot(f\vec{A}) - \vec{A}\cdot(\vec{\nabla}f)$$

$$= \underbrace{\int_V \vec{\nabla}\cdot(f\vec{A})\, d^3r}_{\text{apply Divergence/Gauss's theorem}} - \int_V \vec{A}\cdot(\vec{\nabla}f)\, d^3r$$

$$= \underbrace{\int_S f\vec{A}\cdot d\vec{s}} - \int_V \vec{A}\cdot(\vec{\nabla}f)\, d^3r$$

you only need to know $f(\vec{r})$ and $\vec{A}(\vec{r})$ on the surface (i.e. boundary conditions)

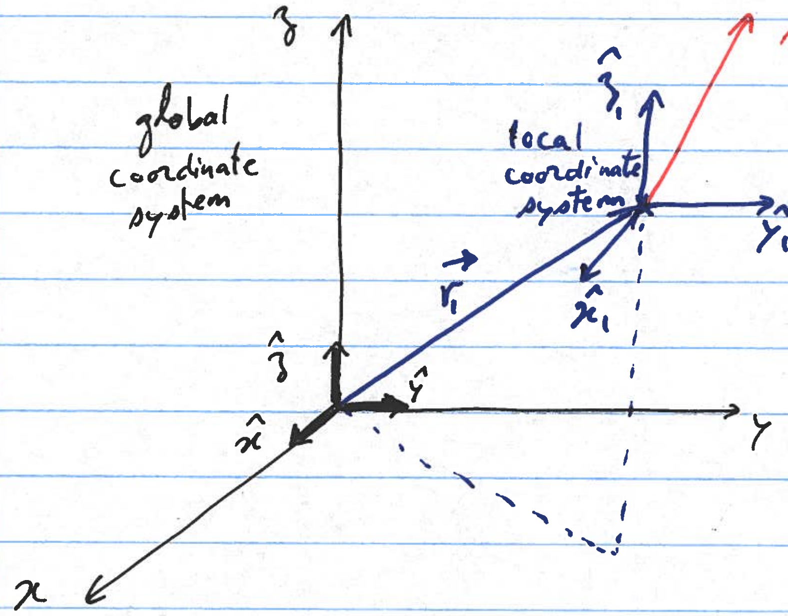
↳ in E&M often fields are zero on a boundary.

<u>note</u>: the same approach can be used ~~by appl~~ to apply Stokes's theorem to certain surface integrals.

# VI Spherical Coordinates (as an example of curvilinear coordinates)

## Reminder about Cartesian Coordinates

$\vec{A}(\vec{r}) = A_x \hat{x}_1 + A_y \hat{y}_1 + A_z \hat{z}_1$




Important: $\hat{x}_1 = \hat{x} = \ldots \hat{x}_n$

$\hat{y}_1 = \hat{y} = \ldots \hat{y}_n$

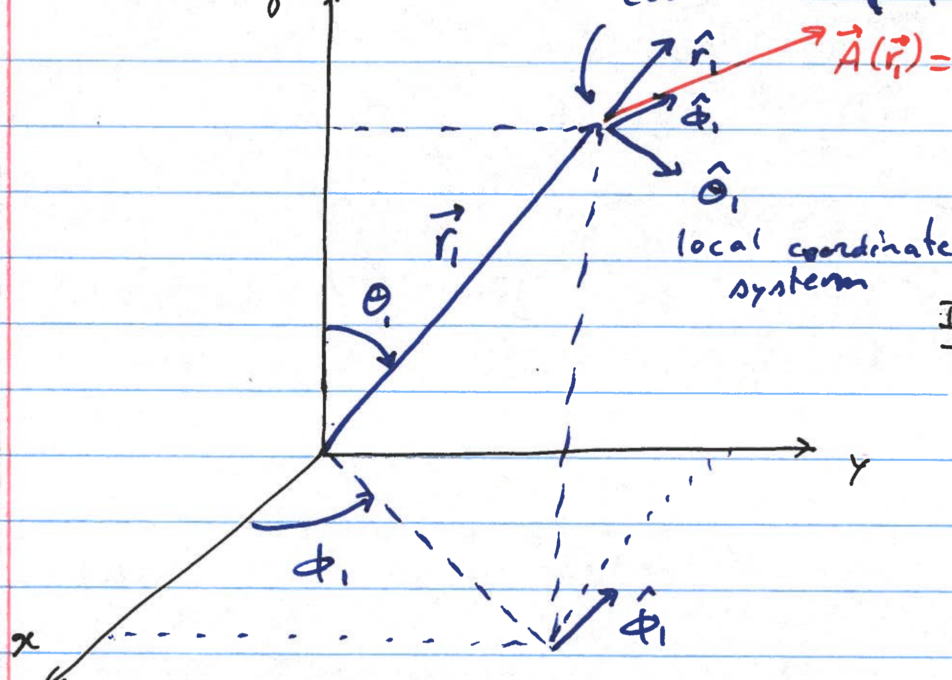
$\hat{z}_1 = \hat{z} = \ldots \hat{z}_n$

(very convenient)

## Spherical Coordinates (global & local)

coordinates $(r_1, \theta_1, \phi_1)$

$\vec{A}(\vec{r}_1) = A_{r_1} \hat{r}_1 + A_{\theta_1} \hat{\theta}_1 + A_{\phi_1} \hat{\phi}_1$




Important:

$\hat{r}_1 \neq \hat{r}_n$

$\hat{\theta}_1 \neq \hat{\theta}_n$

$\hat{\phi}_1 \neq \hat{\phi}_n$

global
coordinate conversion

$$\begin{cases} z = r\cos\theta \\ y = r\sin\theta\sin\phi \\ x = r\sin\theta\cos\phi \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \arctan\left(y/x\right) \end{cases}$$

⚠ The local coordinate system unit vectors depend on $(\theta,\phi)$

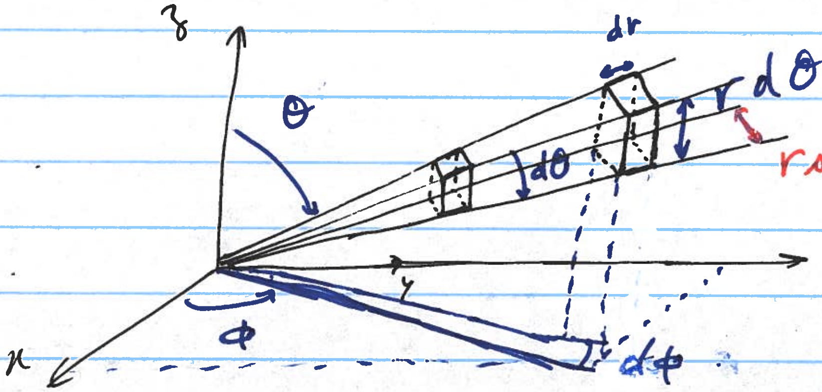
$$\begin{cases} \hat{r}(\theta,\phi) = \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \\ \hat{\theta}(\theta,\phi) = \hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \\ \hat{\phi}(\theta,\phi) = \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{cases}$$

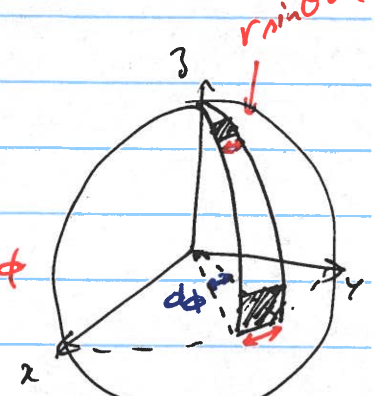
Recommendation: for a complicated problem with spherical symmetry, start in spherical coordinates, but then convert to Cartesian coordinates.

Infinitesimal Volume Integration Element

Cartesian coordinates: $dV = dx\,dy\,dz\,(x_1,y_1,z_1) = dx\,dy\,dz\,(x_n,y_n,z_n)$
↳ "independent" of location.

Spherical Coordinates



⚠ The volume of volume element depends on the coordinates!

$$dV = dr \cdot r\,d\theta \cdot r\sin\theta\,d\phi$$

$$\Longleftrightarrow \boxed{dV = r^2 \sin\theta\,dr\,d\theta\,d\phi} \longrightarrow$$ replace $dx\,dy\,dz$ with this expression

note: $dx\,dy\,dz \neq r^2 \sin\theta\,dr\,d\theta\,d\phi$

## Grad, Div, Curl, etc

Gradient: $$\vec{\nabla} f(\vec{r}) = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

lots of algebra!

$$= \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

depends on location!

for example: $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}$$

$$\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} = \sin\theta\cos\phi$$

also $$\hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$$

etc...