Monday, April 5, 2021

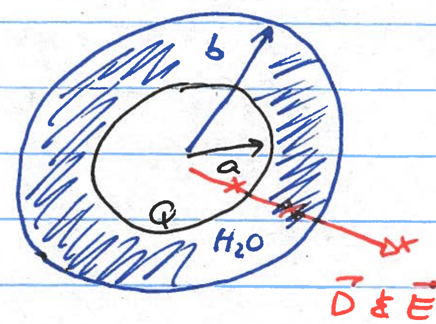
Summary of last time: electric displacement $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

"Gauss's law" $\vec{\nabla} \cdot \vec{D} = \rho_{free}$ $\Big| \; \vec{\nabla} \cdot \vec{E} = \frac{\rho_{free} + \rho_b}{\varepsilon_0}$

Example: Consider a hollow sphere of radius a with a uniform charge Q spread over its surface and surrounded by a thick spherical shell of insulating water/liquid of radius b (thickness $b-a$).

$\rightarrow$ Calculate $\vec{D}$ and $\vec{E}$.

note: $Q = Q_{free}$



for $r<a$:

$\vec{E} = 0$ from Gauss's law

$\vec{P} = 0$ (vacuum/air)

$\Rightarrow \vec{D} = \varepsilon_0 \vec{E} + \vec{P} = 0$

for $r>b$:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r} \quad \text{from Gauss's law}$$

$\vec{P} = 0$ (vacuum/air)

$$\Rightarrow \vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \frac{1}{4\pi} \frac{Q}{r^2} \hat{r}$$

Alternatively, "Gauss's law" for $\vec{D}$ gives

$$\int \vec{D} \cdot d\vec{s} = Q_{free,enclosed} = Q$$

$$\Rightarrow 4\pi r^2 D = Q \Longrightarrow \vec{D} = \frac{1}{4\pi} \frac{Q}{r^2} \hat{r}$$

for $a<r<b$:

"Gauss's law" for $\vec{D}$ gives: $\int \vec{D} \cdot d\vec{s} = Q$

$$\Rightarrow \vec{D} = \frac{1}{4\pi} \frac{Q}{r^2} \hat{r}$$

Since we don't know $\vec{P}$ (yet), we cannot get $\vec{E}$!

Linear Dielectrics

In a linear dielectric (i.e. most materials for small $\vec{E}$), the polarization (dipole moment per volume):

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

where χ_e = electric susceptibility.

note 1: For large $\vec{E}$ there is a deviation from linear behavior:

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} + \underbrace{\varepsilon_0 \chi_e^{(2)} |\vec{E}|^2 \hat{e}}_{\text{responsible for 2nd harmonic generation (e.g. 1064 nm} \to \text{532 nm)}} + \underbrace{\varepsilon_0 \chi_e^{(3)} |\vec{E}|^3 \hat{e}}_{\text{responsible for 4-wave mixing}} + \cdots$$

note 2: In many crystals $\chi_e \longrightarrow \chi_{e_{ij}}$ tensor susceptibility (still linear dielectrics)

thus $P_i = \varepsilon_0 \chi_{e_{ij}} E_j$

$$\Longleftrightarrow \vec{P} = \varepsilon_0 [\chi_e] \vec{E}$$

$\overleftrightarrow{\chi_e}$

If $\vec{P} = \varepsilon_0 \chi_e \vec{E}$, then $\vec{D} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E}$

$$\Leftrightarrow \quad \vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E}$$

$$\Rightarrow \quad \boxed{\vec{D} = \varepsilon \vec{E} \quad \text{with } \varepsilon = \varepsilon_0 (1 + \chi_e)}$$

$\Rightarrow$ "Gauss law": $\boxed{\vec{\nabla} \cdot \vec{E} = \dfrac{\rho_{free}}{\varepsilon}}$

<u>Dielectric constant ε_r</u> (or relative permittivity)

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = 1 + \chi_e = \kappa$$ ← common notation

Examples:

$\varepsilon_{vacuum} = 1$

$\varepsilon_{air} \simeq 1$

$\varepsilon_{glass} \simeq 2.3 - 4.8$ (BK7 → 2.3, Pyrex → 4.8)

$\varepsilon_{diamond} \simeq 5.7 - 5.9$

$\varepsilon_{teflon} \simeq 2.1$, $\varepsilon_{polypropylene} \simeq 2.3$

$\varepsilon_{Si} \simeq 11.7$

$\varepsilon_{AlN} \simeq 8.9 - 9$

$\varepsilon_{GaAs} \simeq 12.4$

⚠ $\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P} = \vec{\nabla} \times (\varepsilon_0 \chi_e \vec{E})$

$$= \varepsilon_0 \chi_e \cancelto{0}{(\vec{\nabla} \times \vec{E})} - \varepsilon_0 \vec{E} \times \vec{\nabla} \chi_e$$

$$= -\varepsilon_0 \vec{E} \times \underbrace{\vec{\nabla} \chi_e}$$

$= 0$ in bulk of material

$\neq 0$ across an interface

Boundary conditions

free surface charge

$$(\vec{D}_1 - \vec{D}_2)_\perp = \sigma_{free}$$

for a linear dielectric $(\vec{D} = \varepsilon \vec{E})$: $\varepsilon_1 \vec{E}_1 - \varepsilon_2 \vec{E}_2 = \sigma_f$

$\vec{E} = -\vec{\nabla} V$, so we get $\boxed{\varepsilon_1 \frac{\partial V}{\partial n} - \varepsilon_2 \frac{\partial V}{\partial n} = \sigma_f}$

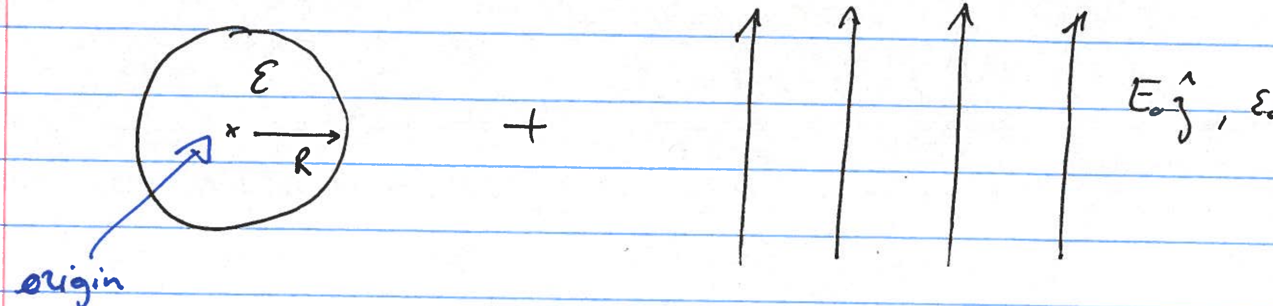
$\hat{n}$ points from "2" into "1"

$\Delta V = -\int \vec{E} \cdot d\vec{l} \Rightarrow$ V is continuous across a boundary.

at boundary $V_1\Big|_{boundary} = V_2\Big|_{boundary}$

Example: Dielectric sphere of radius R in an initially uniform $\vec{E}$-field $\vec{E} = E_0 \hat{z}$

$\longrightarrow$ Calculate $\vec{E}$ everywhere, $\vec{P}$.



note: $\rho_{free} = 0$, $\sigma_{free} = 0$

$\vec{E} = \vec{E}_0 \hat{z} \Rightarrow V = -E_0 z$
$= -E_0 r \cos\theta$

boundary conditions: – at $r=R$
$$\begin{cases} V_{in}(r=R) = V_{out}(r= \\ \varepsilon_0 \frac{\partial V_{out}}{\partial r}\Big|_{r=R} - \varepsilon \frac{\partial V_{in}}{\partial r}\Big|_{r=} \end{cases}$$

– for $r \to \infty$: $V_{out}(r \to \infty) = -E_0 r \cos\theta$

for a linear dielectric $(r<R)$: $\vec{\nabla} \cdot \vec{E} = \frac{\rho_f}{\varepsilon_0} = 0 \Rightarrow \nabla^2 V = 0$ ~~inside dielectric~~

$\Rightarrow$ Laplace's equation applies inside dielectric.

for vacuum $(r>R)$: $\nabla^2 V = 0 \Rightarrow$ Laplace's equation applies everywhere except on boundary surface of sphere.

No ϕ-dependence $\to$ use Legendre polynomials & separation of variables

$$\begin{cases} V_{in}(r,\theta) = \sum_{l=0}^{\infty} a_l \frac{r^l}{R^l} P_l(\cos\theta) \quad \text{for } r<R \\ V_{out}(r,\theta) = \sum_{l=0}^{\infty} \left[b_l \left(\frac{R}{r}\right)^{l+1} + c_l \left(\frac{r}{R}\right)^l \right] P_l(\cos\theta) \quad \text{for } r>R \end{cases}$$

normally, you don't keep this term because it leads to divergences when $r \to \infty$, but in this problem $V(r \to \infty) \to -E_0 r \cos\theta$

boundary conditions : for $r \to \infty$ $\quad V_{out}(r \to \infty) = -E_0 r \cos\theta$

$$b_\ell \left(\frac{R}{r}\right)^{\ell+1} \to 0 \quad \text{for } r \to \infty \quad \Rightarrow \text{no constraints on } b_\ell$$

$$\lim_{r \to \infty} C_\ell \left(\frac{r}{R}\right)^{\ell} P_\ell(\cos\theta) = -E_0 r \cos\theta$$

$$\Rightarrow \text{Since } P_{\ell=1}(\cos\theta) = \cos\theta \Rightarrow \boxed{\begin{cases} C_\ell = 0 & \text{for } \ell \neq 1 \\ C_1 = -E_0 R \end{cases}}$$

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for $r = R$; $\quad V_{in}(r=R) = V_{out}(r=R)$

$$\Leftrightarrow \sum_{\ell=0}^{\infty} a_\ell \cancel{\left(\frac{R}{R}\right)^{\ell}}^{\,1} P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} b_\ell \cancel{\left(\frac{R}{R}\right)^{\ell+1}}^{\,1} P_\ell(\cos\theta) - E_0 R \cancel{\frac{R}{R}} \underbrace{\cos\theta}_{P_1(\cos\theta)}$$

We match the θ-dependence on RHS & LHS.

$$\Rightarrow \quad a_\ell = b_\ell \qquad \text{for } \ell \neq 1 \qquad (1a)$$

$$a_1 = b_1 - E_0 R \qquad \text{for } \ell = 1 \qquad (1b)$$

Also, we know that $\displaystyle \varepsilon \frac{\partial V_{in}}{\partial r}\bigg|_{r=R} = \varepsilon_0 \frac{\partial V_{out}}{\partial r}\bigg|_{r=R}$

$$\varepsilon \sum_{\ell=0}^{\infty} a_\ell \, \ell \, \frac{\cancel{R^{\ell-1}}}{\cancel{R^{\ell}}} P_\ell(\cos\theta) = \varepsilon_0 \sum_{\ell=0}^{\infty} b_\ell (-\ell-1) \frac{\cancel{R^{\ell+1}}}{\cancel{R^{\ell+2}}} P_\ell(\cos\theta) - \varepsilon_0 E_0 \cancel{R} \frac{1}{\cancel{R}} \underbrace{\cos\theta}_{P_1(\cos\theta)}$$

we match the θ-dependence on RHS & LHS :

$$\begin{cases} \varepsilon\, a_\ell\, \ell \, \dfrac{1}{R} = \varepsilon_0\, b_\ell \, \dfrac{(-\ell-1)}{R} & \text{for } \ell \neq 1 \quad \text{(2a)} \\[2ex] \varepsilon\, a_1 (1) \, \dfrac{1}{R} = \varepsilon_0\, b_1 (-1-1) \dfrac{1}{R} - \varepsilon_0 E_0 & \text{for } \ell = 1 \quad \text{(2b)} \end{cases}$$

(1a) & (2a) can only be satisfied if $a_\ell = b_\ell = 0$ for $\ell \neq 1$

$$\Rightarrow \begin{cases} a_1 = b_1 - E_0 R & \text{(1b)} \\[1ex] \dfrac{\varepsilon}{\varepsilon_0} a_1 = -2b_1 - E_0 R & \text{(2b)} \end{cases}$$

$$\Leftrightarrow \frac{\varepsilon}{2\varepsilon_0} a_1 = -b_1 - \frac{E_0 R}{2}$$

$$\text{(1b)} + \text{(2b)} \Rightarrow \left(1 + \frac{\varepsilon}{2\varepsilon_0}\right) a_1 = -E_0 R \overbrace{\left(1 + \frac{1}{2}\right)}^{3/2}$$

$$\Rightarrow \quad a_1 = \frac{-\frac{3}{2} E_0 R}{1 + \frac{\varepsilon}{2\varepsilon_0}}$$

$$\Leftrightarrow \quad \boxed{a_1 = \frac{-3E_0}{2 + \varepsilon/\varepsilon_0} R}$$

↳ plug into (1b): $\dfrac{-3E_0}{2 + \varepsilon/\varepsilon_0} R = b_1 - E_0 R$

$$\Leftrightarrow \quad b_1 = E_0 R \left[\underbrace{1 - \frac{3}{2 + \varepsilon/\varepsilon_0}}_{\frac{2 + \varepsilon/\varepsilon_0 - 3}{2 + \varepsilon/\varepsilon_0}} \right]$$

$$\Leftrightarrow \quad \boxed{b_1 = \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) E_0 R}$$

Thus,

$$V_{in} = \frac{-3E_0}{2 + \varepsilon/\varepsilon_0} \cancel{R} \frac{r}{\cancel{R}} \cos\theta$$

$$\Rightarrow \quad \boxed{V_{in} \; \frac{-3 E_0 r \cos\theta}{2 + \varepsilon/\varepsilon_0} = \frac{-3}{2 + \varepsilon/\varepsilon_0} E_0 z} \quad \text{for } r < R$$

$$V_{out} = \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) E_0 \cancel{R}\, \frac{\cancel{R^2}^{\,3}}{r^2} \cos\theta + \left(-E_0 \cancel{R} \frac{r}{\cancel{R}} \cos\theta \right)$$

$$\Rightarrow \quad \boxed{\begin{aligned} V_{out} &= \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \, \frac{E_0 R^3 \cos\theta}{r^2} - E_0 r \cos\theta \\ &= \left[\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \, \frac{E_0 R^3}{r^3} - E_0 \right] z \end{aligned}} \quad \text{for } r > R$$

<u>note</u>: $\vec{E}_{in} = -\vec{\nabla} V_{in} \Rightarrow \vec{E}_{in} = \dfrac{+3E_0}{2 + \varepsilon/\varepsilon_0} \hat{z}$

$\Rightarrow \vec{E}_{in}$ is a uniform field

$\Rightarrow \vec{P} = \varepsilon_0 \chi_e \vec{E}$ is also uniform

$= (\varepsilon - \varepsilon_0) \vec{E}$

$= \dfrac{3(\varepsilon - \varepsilon_0)}{2 + \varepsilon/\varepsilon_0} E_0 \hat{z}$