Monday, April 12, 2021

Example (continued): Dielectric sphere of radius R in an initially uniform E-field $\vec{E} = E_0 \hat{z}$

$\longrightarrow$ Calculate $\vec{E}$ (or V) everywhere, and $\vec{P}$.

Summary of approach: separation of variables

$$V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell \frac{r^\ell}{R^\ell} P_\ell(\cos\theta) \quad \text{for } r \leq R$$

(inside dielectric sphere)

$$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty} \left[b_\ell \left(\frac{R}{r}\right)^{\ell+1} + c_\ell \left(\frac{r}{R}\right)^\ell \right] P_\ell(\cos\theta) \quad \text{for } r \geq R$$

(outside dielectric sphere)

Boundary conditions:

1_ $V_{out}(r \to +\infty) = -E_0 r \cos\theta \Rightarrow \boxed{\begin{cases} c_{\ell\neq1} = 0 \\ c_{\ell=1} = -E_0 R \end{cases}}$

2_ $V_{in}(r=R) = V_{out}(r=R)$

$$\Rightarrow \boxed{\begin{cases} a_\ell = b_\ell & \text{for } \ell \neq 1 \quad (1a) \\ a_1 = b_1 - E_0 R & \text{for } \ell = 1 \quad (1b) \end{cases}}$$

3_ $$\varepsilon \left.\frac{\partial V_{in}}{\partial r}\right|_{r=R} = \varepsilon_0 \left.\frac{\partial V_{out}}{\partial r}\right|_{r=R}$$

$$\Rightarrow \begin{cases} \varepsilon a_\ell \ell \frac{1}{R} = \varepsilon_0 b_\ell \frac{(-\ell-1)}{R} & \text{for } \ell \neq 1 \quad (2a) \\ \varepsilon a_1 \frac{1}{R} = \varepsilon_0 b_1 \left(-\frac{2}{R}\right) - \varepsilon_0 E_0 & \text{for } \ell = 1 \quad (2b) \end{cases}$$

→ we stopped here last lecture.

(1a) & (2a) can ~~a~~ only be satisfied if $a_\ell = b_\ell = 0$ for $\ell \neq 1$

$$\Rightarrow \begin{cases} a_1 = b_1 - E_0 R & (1b) \\ \frac{\varepsilon}{\varepsilon_0} a_1 = -2b_1 - E_0 R & (2b) \end{cases}$$

$$\Leftrightarrow \frac{\varepsilon}{2\varepsilon_0} a_1 = -b_1 - \frac{E_0 R}{2}$$

$$(1b) + (2b) \Rightarrow \left(1 + \frac{\varepsilon}{2\varepsilon_0}\right) a_1 = \cancel{b_1} - \cancel{b_1} - E_0 R \underbrace{\left(1 + \frac{1}{2}\right)}_{=3/2}$$

$$\Rightarrow a_1 = \frac{-3/2\, E_0 R}{1 + \varepsilon/2\varepsilon_0}$$

$$\Leftrightarrow \boxed{a_1 = \frac{-3 E_0 R}{2 + \varepsilon/\varepsilon_0}}$$

↳ plug into (1b): $\frac{-3E_0 R}{2 + \varepsilon/\varepsilon_0} = b_1 - E_0 R$

$$\Leftrightarrow \quad b_1 = E_0 R \left[\underbrace{1 - \frac{3}{2 + \varepsilon/\varepsilon_0}}_{\frac{2 + \varepsilon/\varepsilon_0 - 3}{2 + \varepsilon/\varepsilon_0}} \right]$$

$$\Leftrightarrow \quad \boxed{b_1 = \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) E_0 R}$$

Thus,

$$V_{in} = \frac{-3E_0}{2 + \varepsilon/\varepsilon_0} \cancel{R} \frac{r}{\cancel{R}} \cos\theta$$

$$\Rightarrow \quad \boxed{V_{in} \; \frac{-3 E_0 r \cos\theta}{2 + \varepsilon/\varepsilon_0} = \frac{-3}{2 + \varepsilon/\varepsilon_0} E_0 z} \quad \text{for } r < R$$

$$V_{out} = \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) E_0 \cancel{R} \frac{\cancel{R^2}^{\,3}}{r^2} \cos\theta + \left(-E_0 \cancel{R} \frac{r}{\cancel{R}} \cos\theta \right)$$

$$\Rightarrow \quad \boxed{\begin{aligned} V_{out} &= \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} E_0 \frac{R^3}{r^2} \cos\theta - E_0 r \cos\theta \\ &= \left[\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \frac{E_0 R^3}{r^3} - E_0 \right] z \end{aligned}} \quad \text{for } r \geq R$$

note: $\vec{E}_{in} = -\vec{\nabla} V_{in} \Rightarrow \vec{E}_{in} = \dfrac{+3E_0}{2+\varepsilon/\varepsilon_0}\,\hat{z}$

$\Rightarrow \vec{E}_{in}$ is a uniform field

$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$

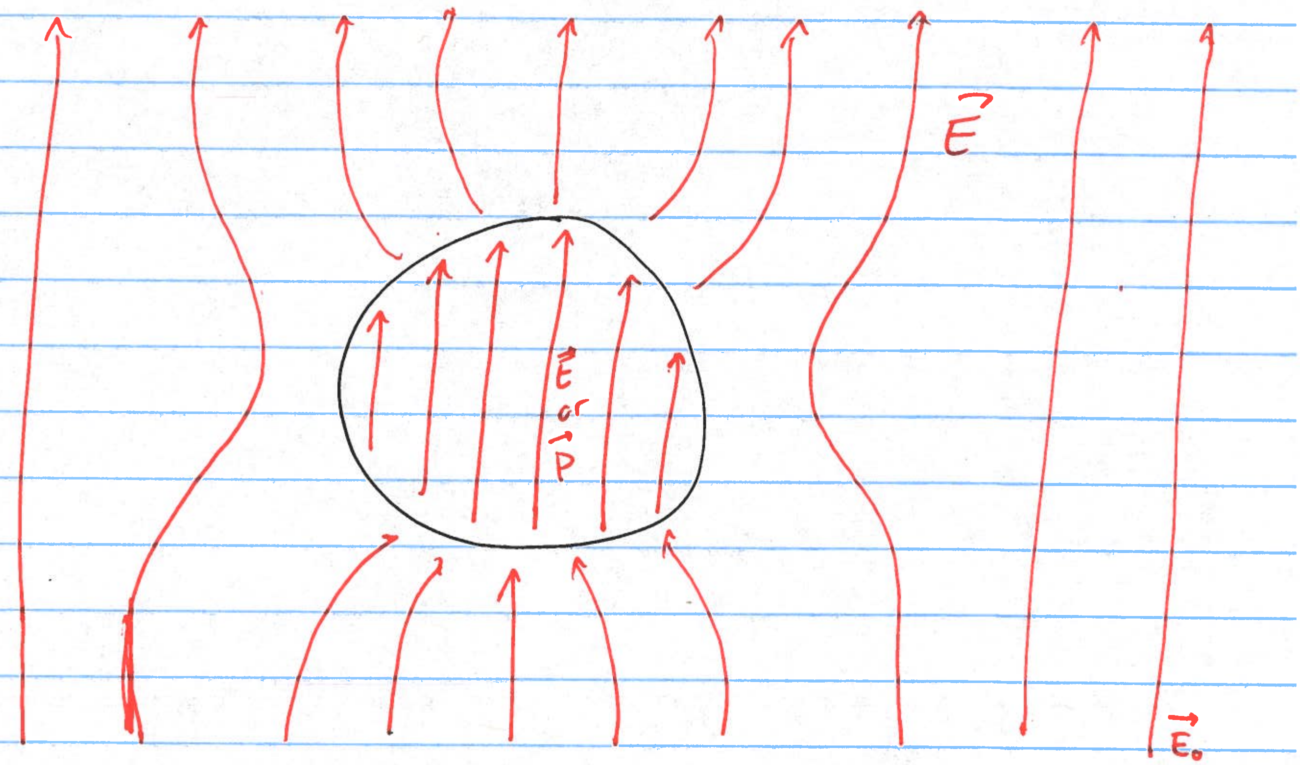
$\Rightarrow \vec{P} = \vec{D} - \varepsilon_0\vec{E}$

$= \varepsilon\vec{E} - \varepsilon_0\vec{E}$

$\Rightarrow \vec{P} = \varepsilon_0 \chi_e \vec{E}$ is also uniform

$= (\varepsilon - \varepsilon_0)\vec{E}$

$= \dfrac{3(\varepsilon-\varepsilon_0)}{2+\varepsilon/\varepsilon_0} E_0\,\hat{z}$ (uniform)



Q: What is the (induced) dipole moment of the dielectric sphere?

A: $\vec{p} = \underbrace{\text{Volume}}_{\frac{4}{3}\pi R^3} \times \vec{P}$

↑ dipole moment per volume

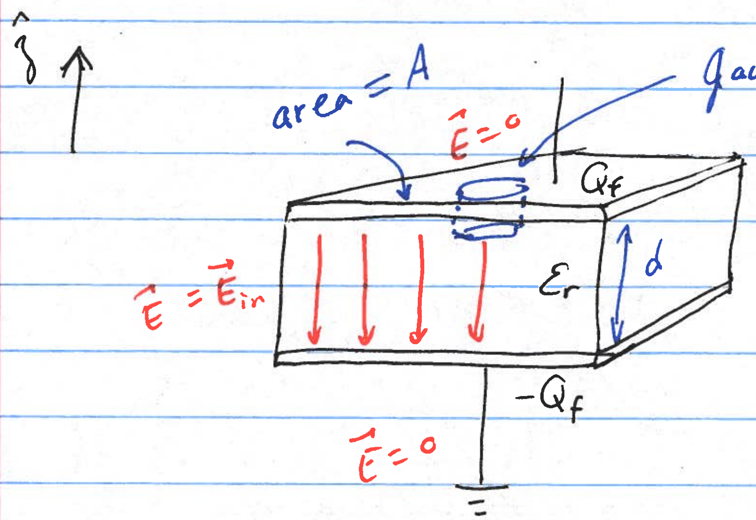
$$\Rightarrow \vec{p} = \frac{4}{\cancel{3}}\pi R^3 \; \frac{\cancel{3}(\varepsilon - \varepsilon_0)}{2 + \varepsilon/\varepsilon_0} \; \underbrace{E_0 \hat{z}}_{\vec{E}}$$

<u>note:</u> If the E-field has a gradient (not part of original solution), then there is a force on the dielectric sphere!

$$\Rightarrow \vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E} = 4\pi R^3 \frac{\varepsilon - \varepsilon_0}{2 + \varepsilon/\varepsilon_0} (\vec{E} \cdot \vec{\nabla})\vec{E}$$

<u>Example</u>: parallel plate capacitor with ~~a~~ linear dielectric between plates.

ε_r = dielectric constant



Gaussian pill box: $EA = \frac{Q_f}{\varepsilon}$

<u>note:</u> ignore/neglect fringing field.

$$\vec{\nabla} \cdot \vec{E} = \rho_f / \varepsilon$$

$$\vec{E}_{in} = -\frac{1}{\varepsilon}\frac{Q_f}{A}\hat{z} = -\frac{\sigma_f}{\varepsilon}\hat{z} \qquad \Rightarrow \Delta V = -\int_{\text{bottom plate}}^{\text{top plate}} \vec{E} \cdot d\vec{z}$$

(reference, $V = 0$)

(with $\varepsilon = \varepsilon_r \varepsilon_0$)

$$= -E_{in} d$$

$$= \frac{1}{\varepsilon} Q_f \frac{d}{A}$$

$$\Rightarrow \Delta V = \frac{d}{\varepsilon A} Q_f \iff \frac{\varepsilon A}{d} = \frac{Q_f}{\Delta V} = \underset{\substack{\uparrow \\ \text{capacitance}}}{C}$$

$$\Rightarrow \boxed{C_{\text{dielectric}} = \frac{\varepsilon A}{d}}$$

<u>note</u>: $$\frac{C_{\text{dielectric}}}{C_{\text{vacuum}}} = \frac{\varepsilon \, A/d}{\varepsilon_0 \, A/d} = \varepsilon_r$$

$$\Rightarrow \boxed{C_{\text{dielectric}} = \varepsilon_r \, C_{\text{vacuum}}}$$

$\Rightarrow$ <u>adding the dielectric increases the capacitance</u>.

Electrostatic Energy in Dielectric systems

In vacuum: $$W = \frac{\varepsilon_0}{2} \int_V \vec{E}^2 \, d^3r$$

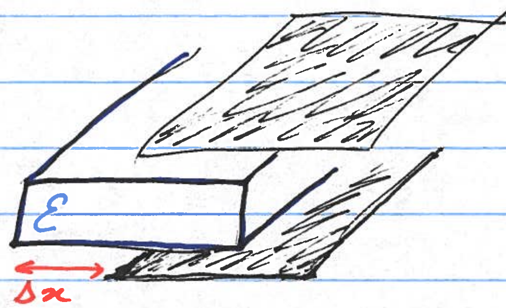
$$\boxed{\begin{gathered} \text{In a dielectric: } W = \frac{1}{2} \int_V \underbrace{\vec{D}}_{\varepsilon \vec{E}} \cdot \vec{E} \, d^3r \\ \Rightarrow W = \frac{\varepsilon}{2} \int \vec{E}^2 \, d^3r \end{gathered}}$$

Example: Force on a dielectric in a capacitor

$$W = \frac{1}{2} C V^2 = \frac{1}{2} C \left(\frac{Q}{C}\right)^2 = \frac{1}{2} \frac{Q^2}{C}$$

$C = \frac{Q}{V} \Leftrightarrow V = \frac{Q}{C}$

Consider a capacitor with a sliding dielectric



$C_\varepsilon = \varepsilon_r C_{\varepsilon_0} \Rightarrow C_\varepsilon > C_{\varepsilon_0}$

$\varepsilon_r > 1$

$\Rightarrow \boxed{W_\varepsilon < W_{\varepsilon_0}}$

for constant Q (~~#~~ different approach needed for constant V)

Capacitor with dielectric has lower energy than capacitor without dielectric

the capacitor will "suck in" the dielectric to lower ~~to~~ ~~to~~ ~~to energy,~~ the energy of the system.

$$\text{Force} = F = -\frac{dW}{dx}$$