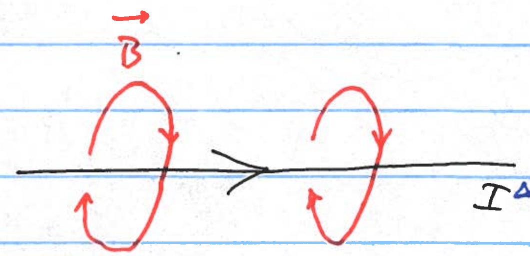
Wednesday, April 14, 2021

Magnetostatics

Magnetic fields are generated by moving charges, i.e. currents.

Ex: Right hand rule gives direction of B-field lines

direction of conventional current (positive charges)



put your thumb along direction of I
↳ fingers give direction of B-field.

magnetic force on a moving charge

$$\vec{F}_{magnetic} = q\,\vec{v} \times \vec{B}$$

⟹ force is $\perp$ to motion $\vec{v}$ and $\vec{B}$.

Force [Newtons]

charge [Coulombs]

velocity [m/s]

magnetic field [Tesla]

note: 1 Tesla = 10^4 Gauss.

Examples: ~~note:~~ magnetic field of the Earth $B_{\substack{Earth\\surface}} \simeq 0.3 - 0.5$ G

Refrigerator/house magnet $B \sim 100$ G (right at surface)

$B_{\text{Jupiter equator}} \simeq 4\ G$, $B_{\text{Sun surface}} \simeq 1\ G$

$B_{\text{sunspot}} \simeq 3000\ G$

near solar system → galactic center

$B_{\text{Milkyway}} \simeq 6 - 40\ \mu G \simeq 10^{-5}\ G$, $B_{\text{neutron star}} = 10^{8-15}\ G$

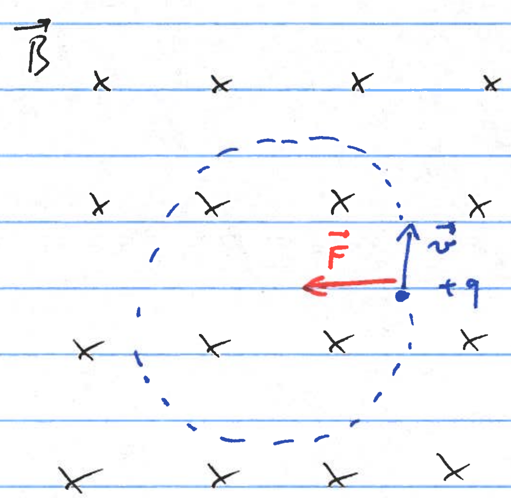
$B_{\text{NMR Lab}} \simeq 17\ T \simeq 170{,}000\ G$

Lorentz Force Law

$$\boxed{\vec{F}_{EM} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)}$$

also works for time-dependent EM fields ~~also~~

Cyclotron Motion



moving charges travel in circular orbits in a magnetic field

[uniform B-field pointing into page]

$F = qvB \Rightarrow a = \frac{qvB}{m}$

For circular motion

$a_c = \frac{v^2}{R}$

$\Rightarrow \frac{qvB}{m} = \frac{v^2}{R}$

$\Leftrightarrow \boxed{R = \frac{mv}{qB}}$

time to complete 1 orbit : $T = \frac{2\pi R}{v} = \frac{2\pi m \cancel{v}}{qB} \Big/ \cancel{v} = \frac{2\pi m}{qB}$

$\Rightarrow$ frequency $= f = \frac{1}{T} = \frac{qB}{2\pi m}$ (Hz)

$\Rightarrow$ $\boxed{\text{Cyclotron frequency} = \omega_c = \frac{qB}{m}}$ rads/s

Magnetic Work : Magnetic Forces do no work

(quantum exceptions)

proof: Work $= W = \vec{F}\cdot\Delta\vec{x} = \int \vec{F}\cdot d\vec{x}$

$= \int \vec{F}_{mag}\cdot d\vec{x}$

$= \int q(\vec{v}\times\vec{B})\cdot \underbrace{d\vec{x}}_{\vec{v}\,dt}$

$= q\int \underbrace{(\vec{v}\times\vec{B})\cdot\vec{v}}_{=0}\, dt$ ($\perp$ to $\vec{v}$ & $\vec{B}$)

$= 0$

note: charge q is independent of ~~rest~~ reference frame.
(different from mass)

[See Griffiths, example 5.3 for a good discussion]

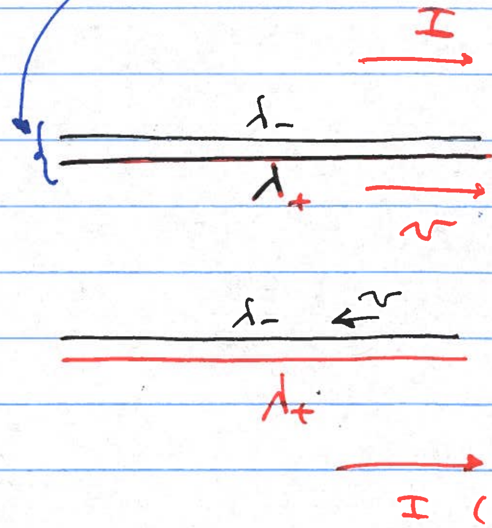
Current & Current Density

Current in a wire

single wire (NOT 2 wires)

model of a wire:
(with conventional current)

real current (e^-):



} wire is always neutral
λ_- cancels λ_+
i.e. $\lambda_- = -\lambda_+$

Current $I = \lambda v$ $\qquad [\frac{C}{m} \times m/s] = [\frac{C}{s}]$

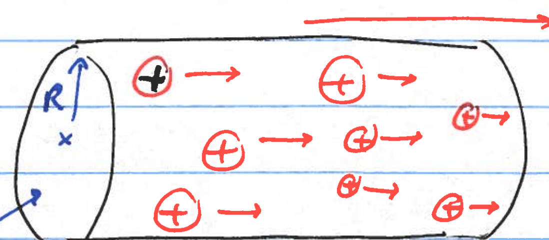
$[\frac{C}{s}]$ = Amps = [A]

note: $1 \text{ Amp} = 1\,A = 1\,C/s = 6.24 \times 10^{18}\, e^-/s$

current density $\vec{J}$

Consider a current of charges flowing uniformly through a wire, or as the ~~[illegible]~~ particle beam in an accelerator.

total current I



Current density = current per unit Area
cross-sectional area

$$\boxed{J = \frac{I}{\text{wire Area}_{\text{cross section}}} = \frac{I}{\pi R^2}}$$

Area $= A$
$= \pi R^2$

formal vector definition: $\boxed{\vec{J} = \rho \vec{v}}$

$\frac{C}{m^2 \cdot s}$ $\quad$ $\frac{C}{m^3}$ $\quad$ m/s

Surface current Density $\vec{K}$

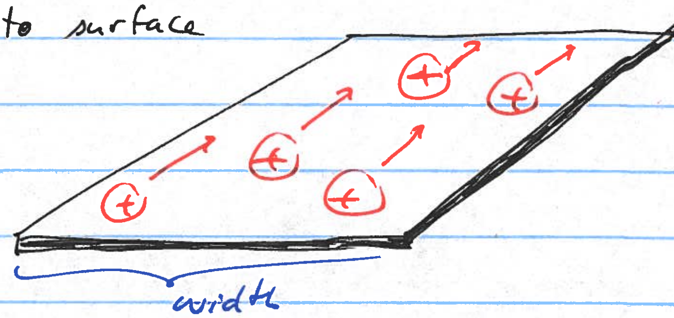
Current confined to surface

$\vec{K} = \sigma \vec{v}$

$\Rightarrow |\vec{K}| = \frac{I}{\text{width}}$

$= [A/m]$

$= \left[\frac{C}{m \cdot s}\right]$



Continuity Equation (conservation of charge)

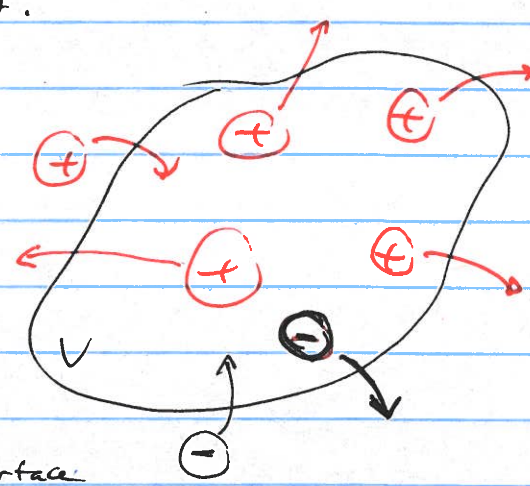
Consider a volume V with surface S with charges flowing out (in) of it.

total current flowing out

$= I_{out} = \int_S \rho \vec{v} \cdot d\vec{s}$

$= \int_S \vec{J} \cdot d\vec{s}$

current flowing across surface



I_{out} is also the total change in charge in V per unit time:

$$I_{out} = -\frac{d}{dt}\underbrace{\int_V \rho \, d^3r}$$

negative for charge flowing out.

thus,

$$-\frac{d}{dt}\int_V \rho \, d^3r = \underbrace{\int_S \vec{J}\cdot d\vec{s}}_{\text{divergence theorem}}$$

$$\int_V \vec{\nabla}\cdot\vec{J}\, d^3r$$

$$\Leftrightarrow \int_V \left[-\frac{d}{dt}\rho\right] d^3r = \int_V (\vec{\nabla}\cdot\vec{J})\, d^3r$$

since this expression (equality) is valid for any volume V including an infinitesimal one, then the integrands must be equal:

$$\boxed{-\frac{d\rho}{dt} = \vec{\nabla}\cdot\vec{J} \Leftrightarrow \frac{d\rho}{dt} + \vec{\nabla}\cdot\vec{J} = 0}$$

continuity equation

or

local conservation of charge.

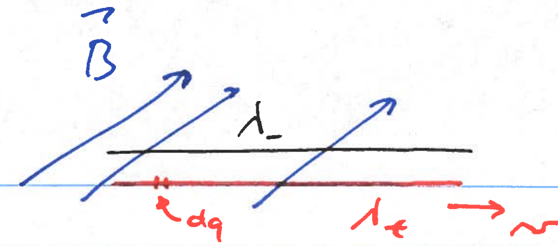
note: if $\frac{d\rho}{dt} = 0$, then $\vec{\nabla}\cdot\vec{J} = 0$ ← magnetostatics

note: local conservation of energy: $-\frac{\partial W}{\partial t} = \vec{\nabla}\cdot\overrightarrow{\text{Power}}$

local conservation of momentum $-\frac{d\vec{P}}{\partial t} = \vec{\nabla}\cdot\underbrace{\overleftrightarrow{\text{stress tensor}}}_{\text{"external forces"}}$

[see chapter 8]

Force on a current



$$\vec{F}_{magnetic} = \int (\vec{v} \times \vec{B})\, dq = \int (\vec{v} \times \vec{B})\, \lambda\, d\ell$$

$$= \int (\underbrace{\lambda \vec{v}}_{\vec{I}} \times \vec{B})\, d\ell$$

$$= \int I\, d\vec{\ell} \times \vec{B} = I \int d\vec{\ell} \times \vec{B}$$

current is constant over wire

$$\Rightarrow \boxed{\vec{F}_{magnetic} = I \int d\vec{\ell} \times \vec{B}}$$

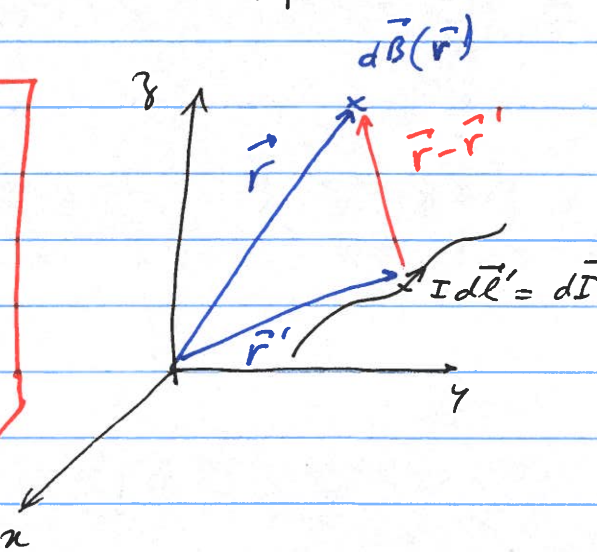
Biot-Savart Law

magnetic field due to an infinitesimal current segment $I\, d\vec{\ell}$:

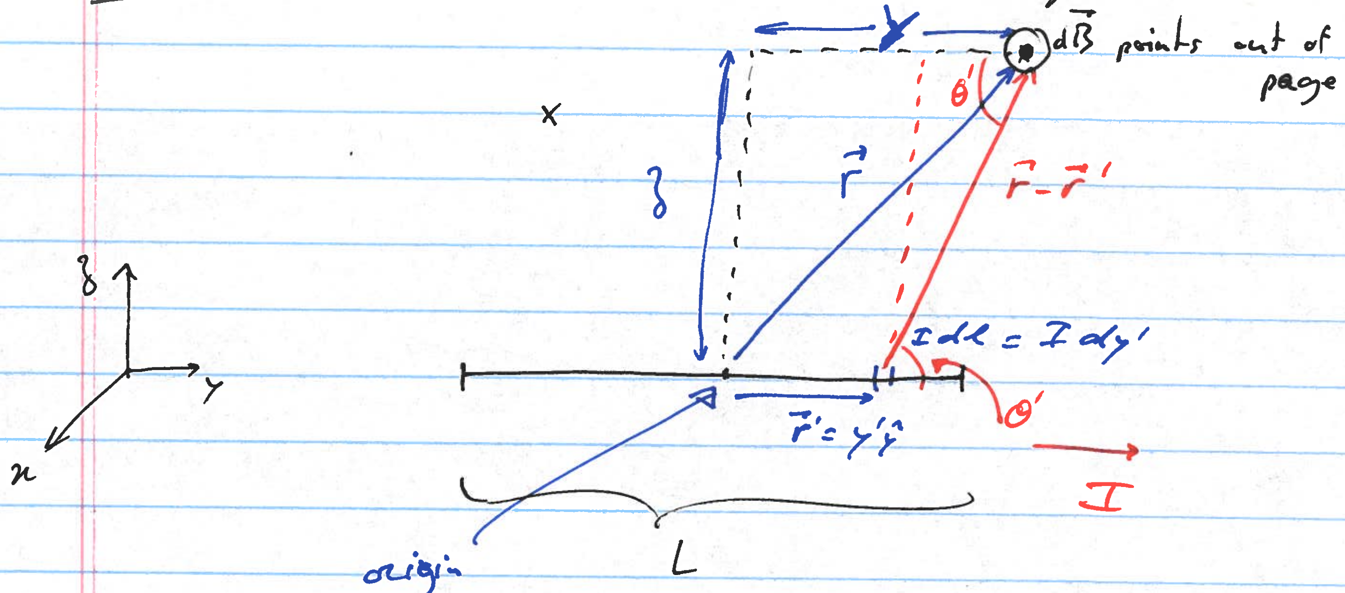
$$d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{I\, d\vec{\ell} \times \hat{r}}{r^2}$$

with $\mu_0 = 4\pi \times 10^{-7}\ N/A^2$
= permeability of free space

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\, d\ell' = \frac{\mu_0}{4\pi} \underbrace{I}_{\text{current/wire length}} \int \frac{d\vec{\ell}\,' \times \widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}}$$



example: magnetic field from a wire segment



$$d\vec{B} = \frac{\mu_0 I}{4\pi} \underbrace{\frac{d\vec{y}\,' \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}}$$

$$\frac{dy'\sin\theta'\,\hat{x}}{|\vec{r}-\vec{r}\,'|^2} = \frac{dy'\,\hat{x}}{(y-y')^2+z^2}\sin\theta'$$

$$= \frac{dy'\,\hat{x}}{(y-y')^2+z^2}\,\frac{z}{\sqrt{(y-y')^2+z^2}}$$

$$= \frac{z\,\hat{x}\,dy'}{\left[(y-y')^2+z^2\right]^{3/2}}$$

$$\Rightarrow \vec{B} = \int_{-L/2}^{L/2} \frac{\mu_0 I}{4\pi}\,\frac{z\,\hat{x}\,dy'}{\left[(y-y')^2+z^2\right]^{3/2}}$$

$$\Leftrightarrow \quad \vec{B} = \frac{\mu_0 I}{4\pi}\, z\, \hat{x} \int_{-L/2}^{L/2} \frac{dy'}{\left[(y-y')^2 + z^2\right]^{3/2}}$$

substitution: $u = y - y' \Rightarrow du = dy'$

$$\Rightarrow \quad \vec{B} = \frac{\mu_0 I}{4\pi}\, z\, \hat{x} \int_{-L/2}^{L/2} \frac{du}{\left(u^2 + z^2\right)^{3/2}}$$

$$= \frac{\mu_0 I}{4\pi}\, z\, \hat{x}\, \frac{1}{z^3} \int_{-L/2}^{L/2} \frac{du}{\left[1 + (u/z)^2\right]^{3/2}}$$

substitution: $v = u/z \Rightarrow dv = \frac{du}{z}$

$$\Rightarrow \quad \vec{B} = \frac{\mu_0 I}{4\pi}\, z\, \hat{x}\, \frac{1}{z^2} \underbrace{\int_{y'=-L/2}^{y'=L/2} \frac{dv}{\left[1 + v^2\right]^{3/2}}}_{\left. \frac{v}{(1+v^2)^{1/2}} \right|_{y'=-L/2}^{y'=L/2}}$$

trigonometric substitution see lecture 6 15 Feb, 2021

$$= \frac{u/z}{\sqrt{1 + (u/z)^2}} = \frac{u}{\sqrt{z^2 + u^2}}$$

$$= \left. \frac{y - y'}{\sqrt{z^2 + (y-y')^2}} \right|_{y'=-L/2}^{y'=L/2}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\hat{x}}{z} \left\{ \frac{y-(L/2)}{\sqrt{z^2+(y-L/2)^2}} - \frac{y+(L/2)}{\sqrt{z^2+(y+L/2)^2}} \right\}$$