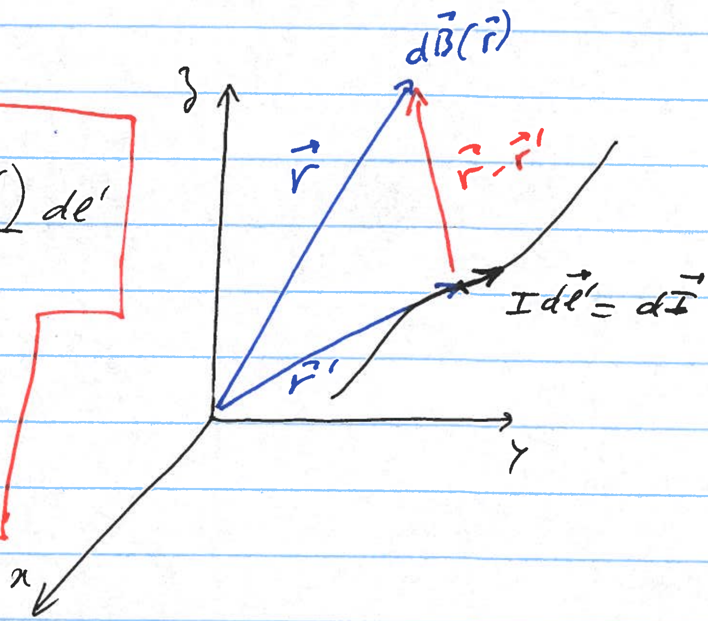
Monday, April 19, 2021

Biot-Savart Law (reminder from last lecture)

The magnetic field due to an infinitesimal current segment $I\,d\vec{\ell}$ is

$$d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I\,d\vec{\ell}\times\hat{r}}{r^2} \qquad \text{with } \mu_0 = 4\pi\times 10^{-7}\ N/A^2$$

$$\boxed{\begin{aligned}\vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\,d\ell' \\ &= \frac{\mu_0 I}{4\pi}\int_{\substack{\text{wire}\\\text{length}}} \frac{d\vec{\ell}\,'\times\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\end{aligned}}$$

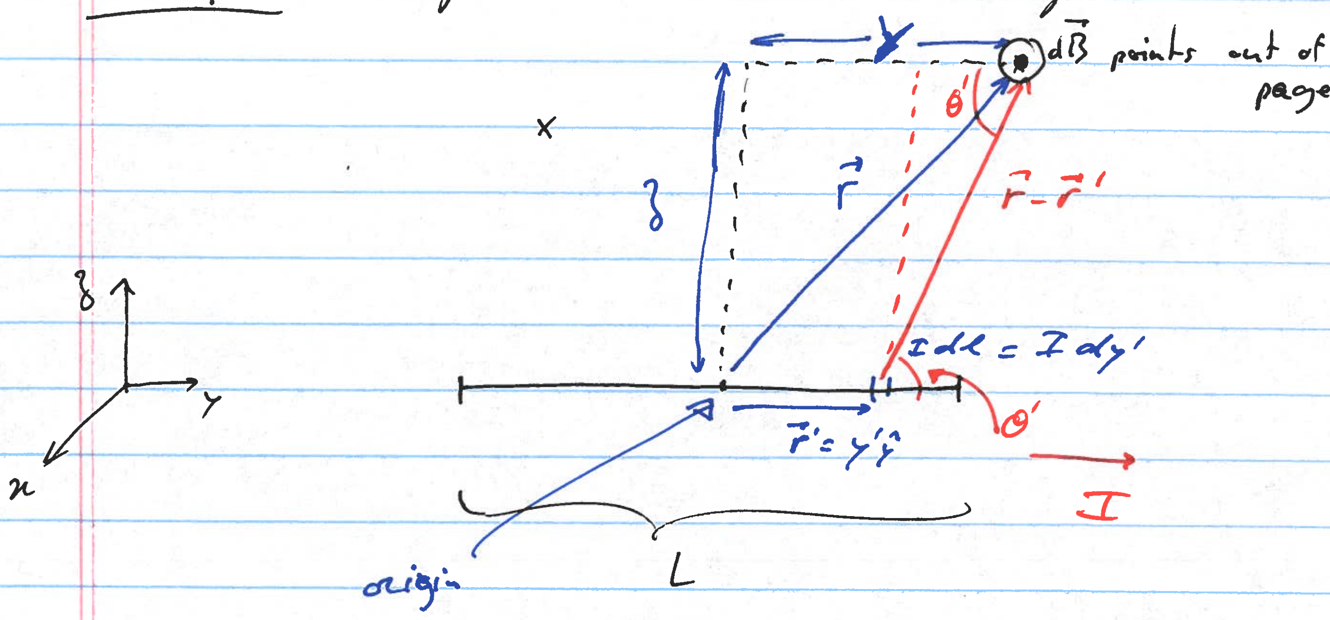


Biot-Savart law

For a volume current: $\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r}\,')\times\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\,d^3r'}$

for a surface current: $\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_S \frac{\vec{K}(\vec{r}\,')\times\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\,ds'}$

example: magnetic field from a wire segment



$$d\vec{B} = \frac{\mu_0 I}{4\pi} \underbrace{\frac{d\vec{y}\,' \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}}$$

$$\frac{dy' \sin\theta' \hat{x}}{|\vec{r}-\vec{r}\,'|^2} = \frac{dy'\hat{x}}{(y-y')^2+z^2} \sin\theta'$$

$$= \frac{dy'\hat{x}}{(y-y')^2+z^2} \frac{z}{\sqrt{(y-y')^2+z^2}}$$

$$= \frac{z\hat{x}\,dy'}{\left[(y-y')^2+z^2\right]^{3/2}}$$

$$\Rightarrow \vec{B} = \int_{-L/2}^{L/2} \frac{\mu_0 I}{4\pi} \frac{z\,\hat{x}\,dy'}{\left[(y-y')^2+z^2\right]^{3/2}}$$

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$$\Leftrightarrow \quad \vec{B} = \frac{\mu_0 I}{4\pi} \, z \, \hat{x} \int_{-L/2}^{L/2} \frac{dy'}{\left[(y-y')^2 + z^2\right]^{3/2}}$$

<u>substitution</u>: $\quad u = y - y' \quad \Rightarrow \quad du = -dy'$

$$\Rightarrow \quad \vec{B} = -\frac{\mu_0 I}{4\pi} \, z \, \hat{x} \int_{-L/2}^{L/2} \frac{du}{\left(u^2 + z^2\right)^{3/2}}$$

$$= -\frac{\mu_0 I}{4\pi} \, z \, \hat{x} \, \frac{1}{z^3} \int_{-L/2}^{L/2} \frac{du}{\left[1 + \left(u/z\right)^2\right]^{3/2}}$$

<u>substitution</u>: $\quad v = u/z \quad \Rightarrow \quad dv = \frac{du}{z}$

$$\Rightarrow \quad \vec{B} = -\frac{\mu_0 I}{4\pi} \, z \, \hat{x} \, \frac{1}{z^2} \underbrace{\int_{y'=-L/2}^{y'=L/2} \frac{dv}{\left[1 + v^2\right]^{3/2}}}$$

trigonometric substitution see Lecture 6 15 Feb, 2021

$$\frac{v}{\left(1 + v^2\right)^{1/2}} \Bigg|_{y'=-L/2}^{y'=L/2}$$

$$= \frac{u/z}{\sqrt{1 + \left(u/z\right)^2}} = \frac{u}{\sqrt{z^2 + u^2}}$$

$$= \frac{y - y'}{\sqrt{z^2 + (y-y')^2}} \Bigg|_{y'=-L/2}^{y'=L/2}$$

$$\vec{B} = -\frac{\mu_0 I}{4\pi}\frac{\hat{x}}{z}\left\{\frac{y-(L/2)}{\sqrt{z^2+(y-L/2)^2}} - \frac{y+(L/2)}{\sqrt{z^2+(y+L/2)^2}}\right\}$$

note: $\lim_{L\to\infty} \vec{B} = -\frac{\mu_0 I}{4\pi}\frac{\hat{x}(-2)}{z} = \frac{\mu_0}{2\pi}\frac{I}{r}\hat{\phi}$

(infinite wire)

generalize

Superposition principle

Magnetic fields obey the superposition principle.

$$\vec{B}_{total}(\vec{r}) = \vec{B}_1(\vec{r}) + \vec{B}_2(\vec{r}) + \dots$$

from source 1 from source 2

Divergence of $\vec{B}$

$$\vec{\nabla}_r\cdot\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \underbrace{\vec{\nabla}_r\cdot\left[\vec{J}(\vec{r}\,')\times\frac{(\widehat{\vec{r}-\vec{r}\,'})}{|\vec{r}-\vec{r}\,'|^2}\right]}_{\text{rule 6 from inside front cover of book}} d^3r'$$

$$= \frac{(\widehat{\vec{r}-\vec{r}\,'})}{|\vec{r}-\vec{r}\,'|^2}\cdot\underbrace{\vec{\nabla}_r\times\vec{J}(\vec{r}\,')}_{=0} - \vec{J}(\vec{r}\,')\cdot\underbrace{\vec{\nabla}_r\times\frac{(\widehat{\vec{r}-\vec{r}\,'})}{|\vec{r}-\vec{r}\,'|^2}}_{=0}$$

no "r" dependence

$\frac{\hat{r}'}{r'^2}$ does not have any rotational circulation see problem 1.63

$$= 0$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

"no magnetic monopoles" law

Curl of $\vec B$

$$\vec\nabla_r \times \vec B(\vec r) = \frac{\mu_0}{4\pi}\int_V \vec\nabla_r \times \left[\frac{\vec J(\vec r\,') \times \widehat{(\vec r - \vec r\,')}}{|\vec r - \vec r\,'|^2}\right] d^3r'$$

formula 8 from inside front cover of book

$$= \left(\frac{\widehat{(\vec r - \vec r\,')}}{|\vec r - \vec r\,'|^2}\cdot \vec\nabla_r\right)\vec J(\vec r\,')\Big|_{=0} - \left(\vec J(\vec r\,')\cdot\vec\nabla_r\right)\frac{\widehat{(\vec r - \vec r\,')}}{|\vec r - \vec r\,'|^2}$$

$$+ \vec J(\vec r\,')\underbrace{\left(\vec\nabla_r\cdot\frac{\widehat{(\vec r - \vec r\,')}}{|\vec r - \vec r\,'|^2}\right)}_{4\pi\delta^3(\vec r - \vec r\,')} - \frac{\widehat{(\vec r - \vec r\,')}}{|\vec r - \vec r\,'|^2}\cancel{\left(\vec\nabla_r\cdot\vec J(\vec r\,')\right)}_{=0}$$

$$= \frac{\mu_0}{4\pi}\left\{\underbrace{\int_V 4\pi\,\delta^3(\vec r - \vec r\,')\,\vec J(\vec r\,')\,d^3r'}_{=4\pi\vec J(\vec r)} - \int_V \left(\vec J(\vec r\,')\cdot\vec\nabla_r\right)\frac{\widehat{(\vec r - \vec r\,')}}{|\vec r - \vec r\,'|^2}\,d^3r'\right\}$$

convert to a surface integral (several lines of vector calculus) involving $\vec J$

but $\vec J$ is zero on surface

$= 0$

$$\Rightarrow \boxed{\vec\nabla \times \vec B(\vec r) = \mu_0 \vec J(\vec r)}$$

Ampère's law

In integral form: $\int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{s} = \int_S \mu_0 \vec{J} \cdot d\vec{s}$

(integrate over a surface S)

$$\underbrace{\int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{s}}_{\text{stoke's theorem}} \qquad \underbrace{\int_S \mu_0 \vec{J} \cdot d\vec{s}}$$

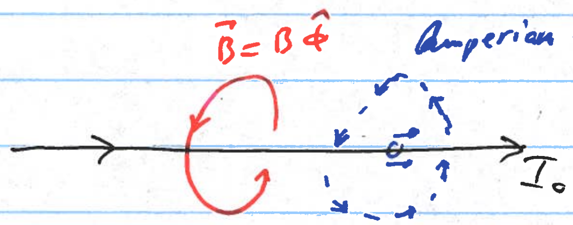
stoke's theorem

$$\oint_{\text{boundary of } S = \text{loop}} \vec{B} \cdot d\vec{l} \qquad \mu_0 \underbrace{\int_S \vec{J} \cdot d\vec{s}}_{I_{enclosed}}$$

$$\Rightarrow \boxed{\oint_{loop} \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}}$$

Ampère's law in integral form

<u>classical example</u>: magnetic field of a wire



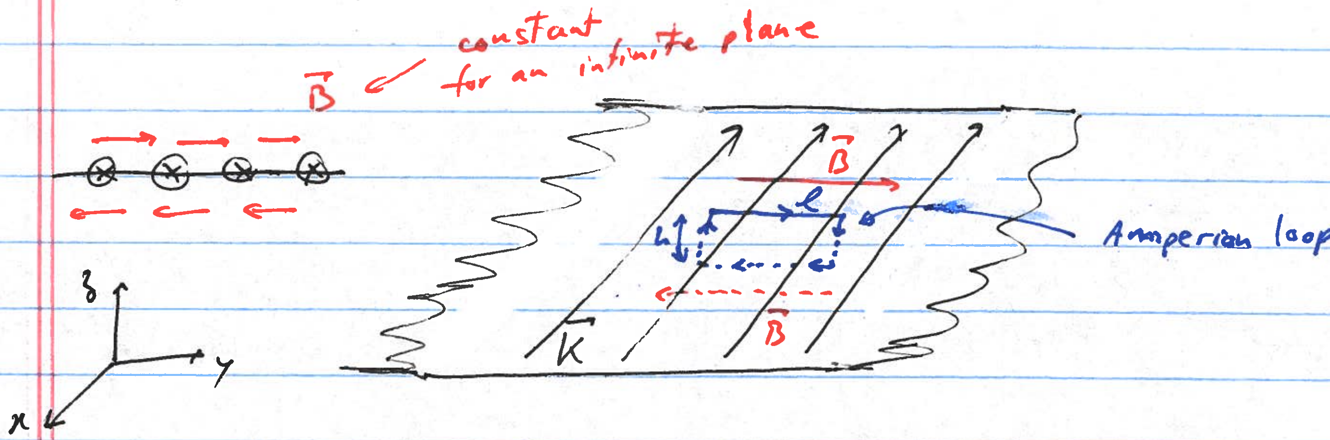
note: use right hand rule to determine if $I_{enclosed}$ is $\begin{cases} \text{positive} \\ \text{negative} \end{cases}$

$$\oint_{loop} \vec{B} \cdot d\vec{l} = \mu I_0 \quad \Rightarrow \quad B \, 2\pi r = \mu_0 I$$

$$\Leftrightarrow \boxed{\vec{B} = \frac{\mu_0}{2\pi} \frac{I}{r} \hat{\phi}}$$

<u>Note</u>: Use Ampère's law whenever there is symmetry.
(otherwise, use Biot-Savart law)

Example: planar surface current



$$\oint_{loop} \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}$$

$$\Leftrightarrow \quad \vec{B}\cdot\vec{l} + \underbrace{\vec{B}\cdot\vec{h}}_{=0} + \vec{B}\cdot\vec{l} + \underbrace{\vec{B}\cdot\vec{h}}_{=0} = \mu_0 K l$$

$\vec{B} \perp \vec{h}$

$$\Leftrightarrow \quad 2Bl = \mu_0 K l \quad \Leftrightarrow \quad \boxed{\begin{cases} \vec{B} = \dfrac{\mu_0 K}{2}\,\hat{y} & \text{above} \\ \vec{B} = -\dfrac{\mu_0 K}{2}\,\hat{y} & \text{below} \end{cases}}$$

B-field is independent of height (for an infinite plane)

Example: Ideal Solenoid ("infinitely" long)

↳ approximate as a series of independent current loops.