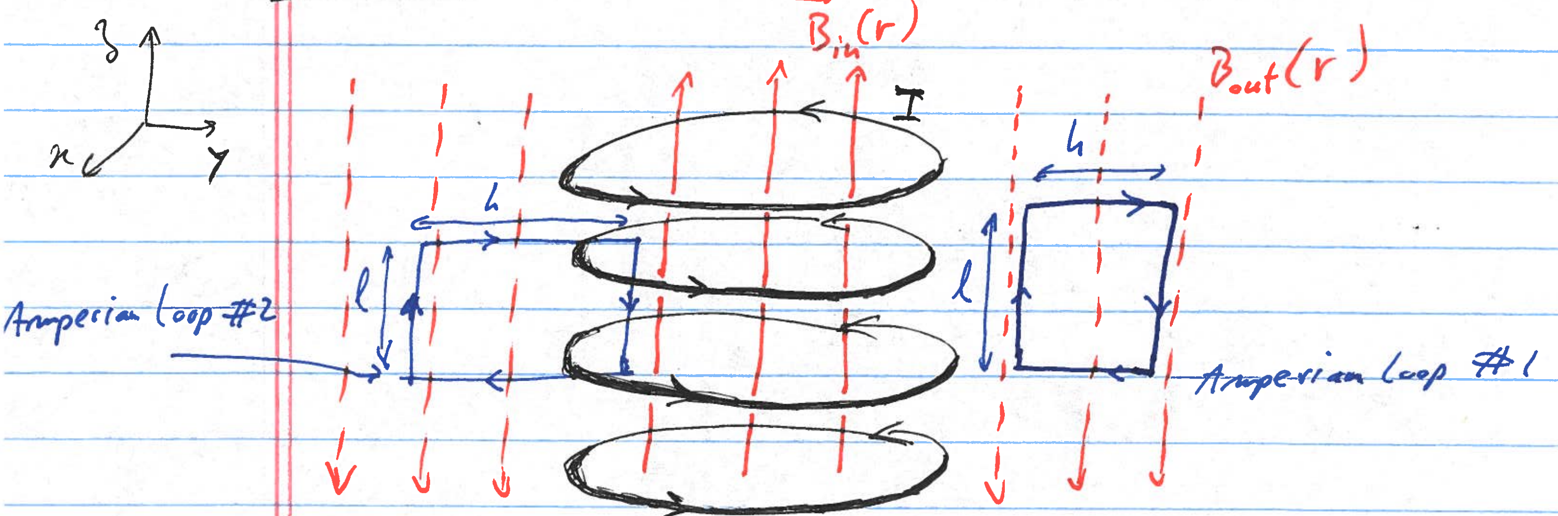
Wednesday, April 21, 2021

Ampère's law (continued):

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} \Leftrightarrow \oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enclosed}$$

Example: Ideal solenoid ("infinitely" long)



[Approximate solenoid as a series of independent current loops]
(each with current I)

Amperian loop #1:

$$\oint_{loop\,1} \vec{B}\cdot d\vec{\ell} = \mu_0 \underbrace{I_{enclosed}}_{=0}$$

$$\Rightarrow \quad -\vec{B}_{close}\cdot\vec{\ell} + \underbrace{\int \vec{B}\cdot d\vec{h}}_{\perp\; =0} + \vec{B}_{far}\cdot\vec{\ell} + \underbrace{\int \vec{B}\cdot d\vec{h}}_{\perp\; =0} = 0$$

$$\Rightarrow \quad \vec{B}_{close} = \vec{B}_{far} \qquad \left(\text{i.e. } \vec{B}_{out}(r_{close}) = \vec{B}_{out}(r_{far})\right)$$

Very far from solenoid, we expect/require: $\vec{B}_{out}(r\to\infty) = 0$

$$\Rightarrow \boxed{\vec{B}_{out}(r) = 0}$$

Amperian Loop #2

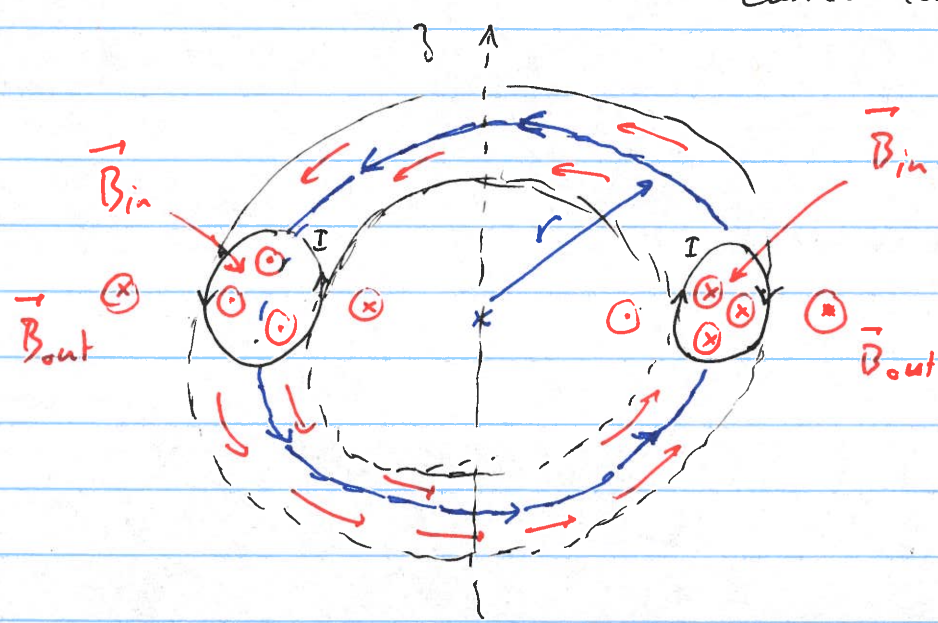
$$\oint_{loop 2} \vec{B}\cdot d\vec{l} = \mu_0 \underbrace{I_{enclosed}}_{-nlI}$$

↑ turns per unit length

$$-B_{in} l + \underbrace{\int \vec{B}\cdot d\vec{l}}_{\perp\ =0} - \underbrace{B_{out}}_{=0} l + \underbrace{\int \vec{B}\cdot d\vec{l}}_{\perp\ =0}$$

$$\Rightarrow -B_{in} l = -\mu_0 n l I \Rightarrow \boxed{\begin{array}{l} \vec{B}_{in} = \mu_0 n I \hat{z} \\ \vec{B}_{out} = 0 \end{array}}$$

Example: Toroidal coil [Approximate as a series of identical current loops in shape a donut/torus]



Amperian loop outside of toroid (r < toroid inner radius or r > toroid outer radius)

$$\oint_{\text{loop out}} \vec{B}\cdot d\vec{l} = \mu_0 \underbrace{I_{enclosed}}_{=0}$$

$$\Rightarrow \quad -B_{out}\, 2\pi r = 0 \quad \Rightarrow \quad \boxed{\vec{B}_{out} = 0}$$

Amperian loop inside of toroid (toroid inner radius < r < toroid outer radius)

$$\oint_{\text{loop in}} \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}$$

total number of turns/loops

$$\Rightarrow \quad B_{in}\, 2\pi r = \mu_0 N I$$

$$\Rightarrow \quad \boxed{\vec{B}_{in} = \frac{\mu_0 N I}{2\pi r}\hat{\phi}}$$

⇒ magnetic is not constant, but it is completely enclosed inside toroid.

(localized magnetic field)

note: – Tokamak fusion "reactors" use Toroidal magnet

– Nuclei have Toroidal currents due to parity violation

↳ nuclear anapole moment

↳ contact interaction for electrons.

Magnetic Vector Potential

reminder: $\vec{\nabla} \times \vec{E} = 0 \implies$ potential V exists such that

$$\vec{E} = -\vec{\nabla} V$$

($\vec{E}$: 3 components; V: 1 components)

Question: Since $\vec{\nabla} \cdot \vec{B} = 0$, is there a corresponding "potential"?

Answer: Yes! The vector potential $\vec{A}(\vec{r})$: $\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$

($\vec{B}$: 3 components; $\vec{A}$: 3 components)

Note: Just like $V(\vec{r})$, $\vec{A}$ is not uniquely defined

$$\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \vec{\nabla}\lambda(\vec{r}) \implies \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times (\vec{\nabla}\lambda(\vec{r}))}_{=0}$$

(λ: random function (well-behaved))

$$\implies \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A}$$

Coulomb gauge: ~~In electrostatics,~~

In magnetostatics, one generally chooses $\vec{A}$ such that

$\boxed{\vec{\nabla} \cdot \vec{A} = 0}$ (definition of Coulomb gauge)

proof that one can always find a $\lambda(\vec{r})$ that will make $\vec{\nabla} \cdot \vec{A} = 0$

- Consider $\vec{A}$ such that $\vec{\nabla} \cdot \vec{A} \neq 0$
- we construct $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$
- Then $\vec{\nabla} \cdot \vec{A}' = \vec{\nabla} \cdot \vec{A} + \underbrace{\vec{\nabla} \cdot \vec{\nabla}\lambda}_{\nabla^2 \lambda}$

If we require $\vec{\nabla} \cdot \vec{A}' = 0$, then $\nabla^2 \lambda = -\vec{\nabla} \cdot \vec{A}$ (scalar "charge distribution")

$\Rightarrow$ we know that we can solve $\nabla^2 \lambda = -\vec{\nabla}\cdot\vec{A}$ since it is identical ~~to~~ mathematically to ~~Laplace~~ poisson's equation.

think of this as "$-\frac{\rho(\vec{r})}{\varepsilon_0}$"

$$\Rightarrow \quad \lambda(\vec{r}) = \frac{1}{4\pi}\int \frac{\vec{\nabla}\cdot\vec{A}(\vec{r}\,')\, d^3r'}{|\vec{r}-\vec{r}\,'|} \quad \left[\text{for } \vec{\nabla}\cdot\vec{A} \to 0 \text{ at } r \to \infty\right]$$

Ampère's law again

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} \iff \underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{A})}_{\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}} = \mu_0\vec{J}$$

from (inside front cover)

$\vec{\nabla}\cdot\vec{A} = 0$ in Coulomb gauge

thus in the Coulomb gauge $\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$

$\Rightarrow$ it's just poisson's equation for each component of $\vec{A}$.

$\hookrightarrow$ All electrostatic techniques for Laplace's equation can be used. !!!

$$\Rightarrow \quad \boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r}\,')\, d^3r'}{|\vec{r}-\vec{r}\,'|}}$$

in coulomb gauge

Example: Calculate $\vec{A}$ for an infinitely long thin wire with current I.

$$\vec{J}(\vec{r}\,')\, d^3r' \doteq I\, d\vec{\ell}\,' = I\, dz'\, \hat{z}$$